

Quiver Hecke Algebras and Categorification

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Abstract. This is a brief introduction to the quiver Hecke algebras of Khovanov, Lauda and Rouquier, emphasizing their application to the categorification of quantum groups. The text is based on lectures given by the author at the ICRA workshop in Bielefeld in August, 2012.

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1. Introduction

Over a century ago Frobenius and Schur understood that the complex representation theory of the symmetric groups S_n for all $n \geq 0$ categorifies the graded Hopf algebra of symmetric functions Pol . The language being used here is much more recent! It means the following. Let $\text{Rep}(\mathbb{C}S_n)$ denote the category of finite dimensional $\mathbb{C}S_n$ -modules and $S(\lambda)$ be the irreducible Specht module indexed by partition $\lambda \vdash n$. The isomorphism classes $\{[S(\lambda)] \mid \lambda \vdash n\}$ give a distinguished basis for Grothendieck group $[\text{Rep}(\mathbb{C}S_n)]$ of this semisimple category. Given an S_m -module V and an S_n -module W , we can form their induction product $V \circ W := \text{Ind}_{S_m \times S_n}^{S_{m+n}} V \boxtimes W$. This operation descends to the Grothendieck groups to give a multiplication making

$$[\text{Rep}(\mathbb{C}S)] := \bigoplus_{n \geq 0} [\text{Rep}(\mathbb{C}S_n)]$$

into a graded algebra. Moreover the restriction functors $\text{Res}_{S_m \times S_n}^{S_{m+n}}$ for all $m, n \geq 0$ induce a comultiplication on $[\text{Rep}(\mathbb{C}S)]$, making it into a graded Hopf algebra. The categorification theorem asserts that it is isomorphic as a graded Hopf algebra to Pol , the canonical isomorphism sending $[S(\lambda)]$ to the *Schur function* $s_\lambda \in Pol$.

There have been many variations and generalizations of this result since then. Perhaps the most relevant for this article comes from the work of Bernstein and Zelevinsky in the late 1970s on the representation theory of the affine Hecke algebra H_n associated to the general linear group $GL_n(F)$ over a non-archimedean local field F (e.g. see [39]). This is even richer algebraically since, unlike $\mathbb{C}S_n$, its finite dimensional representations are no longer completely reducible. Bernstein

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and Zelevinsky showed that the direct sum

$$[\text{Rep}(H)] := \bigoplus_{n \geq 0} [\text{Rep}(H_n)]$$

of the Grothendieck groups of the categories of finite dimensional H_n -modules for all $n \geq 0$ again has a natural structure of graded Hopf algebra. Moreover this graded Hopf algebra is isomorphic to the coordinate algebra $\mathbb{Z}[N]$ of a certain direct limit N of groups of upper unitriangular matrices. In [40], Zelevinsky went on to formulate a p -adic analogue of the Kazhdan-Lusztig conjecture, which was proved by Ginzburg [7]. Zelevinsky's conjecture implies that the basis for $\mathbb{Z}[N]$ arising from the isomorphism classes of irreducible H_n -modules for all $n \geq 0$ coincides with the Lusztig-Kashiwara dual canonical basis. The dual version of this theorem was proved by Ariki in [1], who also investigated certain finite dimensional quotients of H_n called *cyclotomic Hecke algebras*, which categorify the integrable highest weight modules of the corresponding Lie algebra $\mathfrak{g} = \mathfrak{sl}_\infty$. Ariki's work includes the case that the defining parameter of H_n is a primitive p th root of unity, when \mathfrak{sl}_∞ is replaced by the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_p$.

Quiver Hecke algebras were discovered independently in 2008 by Khovanov and Lauda [18, 19] and Rouquier [30]. They are certain Hecke algebras attached to symmetrizable Cartan matrices. It appears that Khovanov and Lauda came upon these algebras from an investigation of endomorphisms of Soergel bimodules (and related bimodules which arise from cohomology of partial flag varieties), while Rouquier's motivation was a close analysis of Lusztig's construction of canonical bases in terms of perverse sheaves on certain quiver varieties. In a perfect analogy with the Bernstein-Zelevinsky theory just described, these algebras categorify the coordinate algebra of the unipotent group N associated to a maximal nilpotent subalgebra of the Kac-Moody algebra \mathfrak{g} arising from the given Cartan matrix.

In fact the picture is even better: the quiver Hecke algebras are naturally \mathbb{Z} -graded, so that the Grothendieck groups of their categories of finite dimensional graded representations are $\mathbb{Z}[q, q^{-1}]$ -modules, with q acting by degree shift. The resulting “graded” Grothendieck groups categorify a $\mathbb{Z}[q, q^{-1}]$ -form for the quantum group \mathbf{f} that is half of the quantized enveloping algebra $U_q(\mathfrak{g})$. Moreover there is an analogue of Ariki's theorem, conjectured originally by Khovanov–Lauda and proved by Varagnolo–Vasserot [32] and Rouquier [31, Corollary 5.8] using geometric methods in the spirit of [7]. There are even cyclotomic quotients of the quiver Hecke algebras which Kang–Kashiwara [13], Rouquier [31, Theorem 4.25] and Webster [35] have used to categorify integrable highest weight modules. Rouquier also observed in (finite or affine) type A that the quiver Hecke algebras become isomorphic to the affine Hecke algebras discussed earlier (at a generic parameter or a root of unity) when suitably localized (see [30, Proposition 3.15] and also [4] in the cyclotomic setting). Thus Ariki's theorem is a special case of the Rouquier–Varagnolo–Vasserot categorification theorem just mentioned (see [5]).

Even more variations on the quiver Hecke algebras have subsequently emerged, including a twisted version related to affine Hecke algebras of type B introduced by Varagnolo and Vasserot [33], and the quiver Hecke superalgebras of Kang, Kashiwara and Tsuchioka [16]. The latter superalgebras generalize Wang's spin Hecke

algebras [34] and the odd nil Hecke algebra of Ellis, Khovanov and Lauda [10], and give a completely new “supercategorification” of the same quantum groups/highest weight modules as above (see [15]). We also mention the work [14] which connects quiver Hecke algebras to quantum affine algebras, potentially providing a direct algebraic link between the categorifications of $\mathbb{Z}[N]$ arising via quiver Hecke algebras and the ones introduced by Hernandez and Leclerc in [12].

For the future perhaps the most exciting development arising from these new algebras is the introduction again by Khovanov-Lauda [20] and Rouquier [31] of certain 2-categories called *2-Kac-Moody algebras*. These categorify Lusztig’s idempotent version $\dot{U}_q(\mathfrak{g})$ of the quantized enveloping algebra of \mathfrak{g} (see [20, 35, 37]). In the case $\mathfrak{g} = \mathfrak{sl}_2$ this goes back to work of Chuang-Rouquier [8] and Lauda [24]. In the introduction of [30], Rouquier promises to define a tensor product on the 2-category of dg 2-representations of the 2-Kac-Moody algebra, the ultimate goal being to construct 4-dimensional TQFTs in fulfillment of predictions made long ago by Crane and Frenkel [9]. Webster has also suggested a more down-to-earth diagrammatic approach to constructing categorifications of tensor products of integrable highest weight modules in finite types in [35, 36].

In this article we will not discuss at all any of these higher themes, aiming instead to give a gentle and self-contained introduction to the quiver Hecke algebras and their connection to Lusztig’s algebra \mathbf{f} , focussing just on the case of symmetric Cartan matrices for simplicity. In the last section of the article we specialize further to finite type and explain some of the interesting homological properties of quiver Hecke algebras in that setting, similar in spirit to those of a quasi-hereditary algebra, despite being infinite-dimensional. As we go we have included proofs or sketch proofs of many of the foundational results, before switching into full survey mode later on. To improve readability, references to the literature are deferred to the end of each section.

2. Quiver Hecke algebras

In this opening section, we give a general introduction to the definition and structure of quiver Hecke algebras.

Gradings. Fix once and for all an algebraically closed ground field \mathbb{K} . Everything (vector spaces, algebras, modules, ...) will be \mathbb{Z} -graded. For a graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$, its graded dimension is $\text{Dim } V := \sum_{n \in \mathbb{Z}} (\dim V_n) q^n$, where q is a formal variable. Of course this only makes sense if V is *locally finite dimensional*, i.e. the graded pieces of V are finite dimensional. Typically V will also be *bounded below*, i.e. $V_n = 0$ for $n \ll 0$, in which case $\text{Dim } V$ is a formal Laurent series in q . We write $q^m V$ for the upward degree shift by m steps, so $q^m V$ is the graded vector space with $(q^m V)_n := V_{n-m}$, and then $\text{Dim } q^m V = q^m \text{Dim } V$. More generally, for $f(q) = \sum_{n \in \mathbb{Z}} f_n q^n$, we write $f(q)V$ for $\bigoplus_{n \in \mathbb{Z}} (q^n V)^{\oplus n}$. Finally we write $\text{Hom}(V, W)$ for the graded vector space $\bigoplus_{n \in \mathbb{Z}} \text{Hom}(V, W)_n$, where $\text{Hom}(V, W)_n$ denotes the linear maps $f : V \rightarrow W$ that are homogeneous of degree n , i.e. they map V_m into W_{m+n} . Note then that $\text{End}(V) := \text{Hom}(V, V)$ is a graded algebra.

Demazure operators. Recall that the symmetric group S_n is generated by the basic transpositions t_1, \dots, t_{n-1} subject only to the braid relations $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ and $t_i t_j = t_j t_i$ for $|i - j| > 1$, plus the quadratic relations $t_i^2 = 1$. The *length* $\ell(w)$ of $w \in S_n$ is $\#\{1 \leq i < j \leq n \mid w(i) > w(j)\}$. We will denote the longest element of S_n by $w_{[1,n]}$. This is the permutation $1 \mapsto n, 2 \mapsto n-1, 3 \mapsto n-2, \dots, n \mapsto 1$, which is of length $\frac{1}{2}n(n-1)$. Letting $[n] := \frac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]! := [n][n-1] \cdots [2][1]$, we have the well-known factorization of the Poincaré polynomial of S_n :

$$\sum_{w \in S_n} q^{2\ell(w)} = q^{\frac{1}{2}n(n-1)}[n]!. \quad (2.1)$$

Let S_n act on the polynomial algebra $Pol_n := \mathbb{K}[x_1, \dots, x_n]$ by permuting the variables. Viewing Pol_n as a graded algebra with each x_i in degree 2, this is an action by graded algebra automorphisms. So the invariants form a graded subalgebra $Sym_n := Pol_n^{S_n}$, namely, the algebra of *symmetric polynomials*. This is again a free polynomial algebra generated by the elementary symmetric polynomials $e_r := \sum_{1 \leq i_1 < \dots < i_r \leq n} x_{i_1} \cdots x_{i_r}$ for $r = 1, \dots, n$, hence

$$\text{Dim } Sym_n = \frac{1}{(1 - q^2)(1 - q^4) \cdots (1 - q^{2n})}. \quad (2.2)$$

For $i = 1, \dots, n-1$, we have the *Demazure operator*

$$\partial_i : Pol_n \rightarrow Pol_n, \quad f \mapsto \frac{t_i(f) - f}{x_i - x_{i+1}}. \quad (2.3)$$

This is a homogeneous linear map of degree -2 such that $\partial_i(fg) = \partial_i(f)g + t_i(f)\partial_i(g)$. From this identity, it is easy to see that ∂_i is a Sym_n -module homomorphism. The endomorphisms $\partial_1, \dots, \partial_{n-1}$ satisfy the same braid relations as in the symmetric group, hence for each $w \in S_n$ there is a well-defined operator $\partial_w \in \text{End}(Pol_n)_{-2\ell(w)}$ such that $\partial_w = \partial_{i_1} \cdots \partial_{i_k}$ if $w = t_{i_1} \cdots t_{i_k}$ is a *reduced expression* for w , i.e. $k = \ell(w)$. Moreover we have that $\partial_i^2 = 0$ for each $i = 1, \dots, n-1$.

Theorem 2.1. *The polynomial algebra Pol_n is a free Sym_n -module of rank $n!$, with basis $(b_w)_{w \in S_n}$ defined from $b_w := \partial_w(x_2 x_3^2 \cdots x_n^{n-1})$. Each b_w is homogeneous of degree $n(n-1) - 2\ell(w)$, and $b_{w_{[1,n]}} = 1$.*

Proof. We first show by induction on n that $b_{w_{[1,n]}} = 1$. Let $w_{[2,n]}$ denote the longest element of S_{n-1} embedded into S_n as the permutations fixing 1, so that $w_{[1,n]} = t_{n-1} \cdots t_1 w_{[2,n]}$. Then:

$$\begin{aligned} \partial_{w_{[1,n]}}(x_2 x_3^2 \cdots x_n^{n-1}) &= \partial_{n-1} \cdots \partial_1 \partial_{w_{[2,n]}}((x_2 \cdots x_n)(x_3 \cdots x_n^{n-2})) \\ &= \partial_{n-1} \cdots \partial_1 (x_2 \cdots x_n \partial_{w_{[2,n]}}(x_3 \cdots x_n^{n-2})) = 1. \end{aligned}$$

Now we use this to show that the elements $(b_w)_{w \in S_n}$ are Sym_n -linearly independent. Suppose that $\sum_{w \in S_n} p_w b_w = 0$ for some $p_w \in Sym_n$, not all of which are zero. Let $w \in S_n$ be of minimal length such that $p_w \neq 0$, and write $w_{[1,n]} = w'w$

for $w' \in S_n$. Then apply $\partial_{w'}$ to the identity $\sum_{w \in S_n} p_w \partial_w (x_2 x_3^2 \cdots x_n^{n-1}) = 0$ to deduce that $p_w = 0$, a contradiction. Finally to complete the proof we check graded dimensions:

$$\text{Dim} \left(\bigoplus_{w \in S_n} \text{Sym}_n b_w \right) \stackrel{(2.2)}{=} \frac{\sum_{w \in S_n} q^{n(n-1)-2\ell(w)}}{(1-q^2)(1-q^4) \cdots (1-q^{2n})} \stackrel{(2.1)}{=} \frac{1}{(1-q^2)^n} = \text{Dim } \text{Pol}_n.$$

□

Corollary 2.2. *The endomorphism algebra $\text{End}_{\text{Sym}_n}(\text{Pol}_n)$ is isomorphic to the algebra of $n! \times n!$ matrices with entries in Sym_n . More precisely, its center is identified with Sym_n , it is free as a module over its center with a basis of matrix units $(e_{x,y})_{x,y \in S_n}$ defined from $e_{x,y}(b_w) := \delta_{y,w} b_x$, and each $e_{x,y}$ is homogeneous of degree $2(\ell(y) - \ell(x))$.*

The nil Hecke algebra. The nil Hecke algebra NH_n is the quiver Hecke algebra (to be defined formally in the next subsection) for the trivial quiver \bullet . By definition, it is the associative graded \mathbb{K} -algebra with homogeneous generators x_1, \dots, x_n of degree 2 and $\tau_1, \dots, \tau_{n-1}$ of degree -2 , subject to the following relations: the x_i 's commute, the τ_i 's satisfy the same braid relations as in the symmetric group plus the quadratic relations $\tau_i^2 = 0$, and finally $\tau_i x_j - x_{t_i(j)} \tau_i = \delta_{i+1,j} - \delta_{i,j}$. As the τ_i 's satisfy the braid relations, there are well-defined elements $\tau_w \in NH_n$ for each $w \in S_n$ such that $\tau_w = \tau_{i_1} \cdots \tau_{i_k}$ whenever $w = t_{i_1} \cdots t_{i_k}$ is a reduced expression. The definition of NH_n ensures that we can make the polynomial algebra Pol_n into a left NH_n -module by declaring that each x_i acts by left multiplication and each τ_i acts by the Demazure operator ∂_i .

Theorem 2.3. *The nil Hecke algebra NH_n has basis*

$$\{x_1^{m_1} \cdots x_n^{m_n} \tau_w \mid w \in S_n, m_1, \dots, m_n \geq 0\}.$$

Moreover the action of NH_n on Pol_n induces a graded algebra isomorphism

$$NH_n \xrightarrow{\sim} \text{End}_{\text{Sym}_n}(\text{Pol}_n).$$

Proof. It is clear from the relations that the given monomials span NH_n . To show that they are linearly independent, suppose there is a non-trivial linear relation $\sum_{w \in S_n} \sum_{m_1, \dots, m_n \geq 0} c_{w, m_1, \dots, m_n} x_1^{m_1} \cdots x_n^{m_n} \tau_w = 0$. Let w be of minimal length such that $c_{w, m_1, \dots, m_n} \neq 0$ for some m_1, \dots, m_n . Write $w_{[1,n]} = ww'$ then act on the vector $b_{w'}$ from Theorem 2.1 to obtain the desired contradiction. This argument shows in fact that the homomorphism $NH_n \rightarrow \text{End}_{\text{Sym}_n}(\text{Pol}_n)$ induced by the action of NH_n on Pol_n is injective. Finally it is surjective by a graded dimension calculation. □

The theorem shows in particular that the algebra Pol_n embeds into NH_n as the subalgebra generated by x_1, \dots, x_n . Using also Corollary 2.2, we deduce that

$Z(NH_n) = Sym_n$, and NH_n is isomorphic to the algebra of $n! \times n!$ matrices over its center. Let

$$e_n := x_2 x_3^2 \cdots x_n^{n-1} \tau_{w_{[1,n]}}. \quad (2.4)$$

Recalling Theorem 2.1, we have that $e_n b_1 = b_1$ and $e_n b_w = 0$ for $w \neq 1$. Hence e_n corresponds under the isomorphism from Theorem 2.3 to the matrix unit $e_{1,1}$ of Corollary 2.2, so it is a primitive idempotent. It follows that

$$P_n := q^{\frac{1}{2}n(n-1)} NH_n e_n \quad (2.5)$$

is an indecomposable projective module. It has a unique irreducible *graded* quotient which we denote by L_n .

Corollary 2.4. *The left regular module NH_n is isomorphic to $[n]!P_n$ as a graded module. Hence $\text{Dim } L_n = [n]!$.*

Proof. Using Theorem 2.3, we identify NH_n with $\text{End}_{Sym_n}(Pol_n)$. Then as in Corollary 2.2 we have that $NH_n = \bigoplus_{w \in S_n} NH_n e_{w,w}$. Right multiplication by $e_{w,1}$ is an isomorphism of graded modules $NH_n e_{w,w} \cong q^{2\ell(w)} NH_n e_{1,1}$. Thus

$$NH_n \cong \bigoplus_{w \in S_n} q^{2\ell(w)} NH_n e_n \stackrel{(2.1)}{\cong} [n]! \left(q^{\frac{1}{2}n(n-1)} NH_n e_n \right) = [n]!P_n.$$

Finally $\text{Dim } L_n = \text{Dim } \text{Hom}_{NH_n}(NH_n, L_n) = [n]! \text{Dim } \text{Hom}_{NH_n}(P_n, L_n) = [n]!$. \square

Corollary 2.5. $P_n \cong q^{-\frac{1}{2}n(n-1)} Pol_n$.

Proof. It is clear from Theorems 2.3 and 2.1 that Pol_n is a projective indecomposable NH_n -module, so it is isomorphic to P_n up to a degree shift. To determine the degree shift, compare graded dimensions. \square

The quiver Hecke algebra. Fix now a loop-free quiver with finite vertex set I . For $i, j \in I$, let $m_{i,j}$ denote the number of directed edges $i \rightarrow j$. The corresponding (symmetric) *Cartan matrix* $C = (c_{i,j})_{i,j \in I}$ is defined from $c_{i,i} := 2$ and $c_{i,j} := -m_{i,j} - m_{j,i}$ for $i \neq j$. To C there is an associated (symmetric) Kac-Moody algebra \mathfrak{g} . We fix a choice of root datum for \mathfrak{g} . This gives us a *weight lattice* P , which is a finitely generated free abelian group equipped with a symmetric bilinear form $P \times P \rightarrow \mathbb{Q}, (\lambda, \mu) \mapsto \lambda \cdot \mu$, containing *simple roots* $(\alpha_i)_{i \in I}$ and *fundamental weights* $(\varpi_i)_{i \in I}$ such that $\alpha_i \cdot \alpha_j = c_{i,j}$ and $\alpha_i \cdot \omega_j = \delta_{i,j}$ for all $i, j \in I$. The *root lattice* is $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset P$. Also let $Q^+ := \bigoplus_{i \in I} \mathbb{N}\alpha_i \subset Q$, and define the *height* of $\alpha = \sum_{i \in I} c_i \alpha_i \in Q^+$ to be the sum $\text{ht}(\alpha) := \sum_{i \in I} c_i$ of its coefficients. Finally let $\langle I \rangle$ denote the set of all words in the alphabet I and, for $\alpha \in Q^+$ of height n , let $\langle I \rangle_\alpha \subset \langle I \rangle$ denote the words $\mathbf{i} = i_1 \cdots i_n \in \langle I \rangle$ such that $\alpha_{i_1} + \cdots + \alpha_{i_n} = \alpha$. The symmetric group S_n acts on $\langle I \rangle_\alpha$ by permuting letters in the obvious way.

Let $q_{i,j}(u, v) \in \mathbb{K}[u, v]$ denote 0 if $i = j$ or $(v - u)^{m_{i,j}}(u - v)^{m_{j,i}}$ if $i \neq j$. For $\alpha \in Q^+$ of height n , the *quiver Hecke algebra* H_α is the associative \mathbb{K} -algebra on generators $\{1_{\mathbf{i}} \mid \mathbf{i} \in \langle I \rangle_\alpha\} \cup \{x_1, \dots, x_n\} \cup \{\tau_1, \dots, \tau_{n-1}\}$ subject to the following relations:

- ▷ the 1_i 's are orthogonal idempotents summing to the identity $1_\alpha \in H_\alpha$;
- ▷ $1_i x_k = x_k 1_i$ and $1_i \tau_k = \tau_k 1_{t_k(i)}$;
- ▷ x_1, \dots, x_n commute;
- ▷ $(\tau_k x_l - x_{t_k(l)} \tau_k) 1_i = \delta_{i_k, i_{k+1}} (\delta_{k+1, l} - \delta_{k, l}) 1_i$;
- ▷ $\tau_k^2 1_i = q_{i_k, i_{k+1}} (x_k, x_{k+1}) 1_i$;
- ▷ $\tau_k \tau_l = \tau_l \tau_k$ if $|k - l| > 1$;
- ▷ $(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) 1_i = \delta_{i_k, i_{k+2}} \frac{q_{i_k, i_{k+1}}(x_k, x_{k+1}) - q_{i_k, i_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} 1_i$.

There is a well-defined \mathbb{Z} -grading on H_α such that each 1_i is of degree 0, each x_j is of degree 2, and each $\tau_k 1_i$ is of degree $-\alpha_{i_k} \cdot \alpha_{i_{k+1}}$.

Note right away that if $\alpha = n\alpha_i$ for $i \in I$, then the quiver Hecke algebra $H_{n\alpha_i}$ is just a copy of the nil Hecke algebra NH_n . In particular there is just one irreducible graded left $H_{n\alpha_i}$ -module up to isomorphism and degree shift. A representative for it may be constructed as the irreducible head $L(i^n)$ of the projective indecomposable module

$$P(i^n) := q^{\frac{1}{2}n(n-1)} H_{n\alpha_i} e_n \quad (2.6)$$

where $e_n \in H_{n\alpha_i}$ is the primitive idempotent defined like in (2.4). By Corollary 2.4, $L(i^n)$ has graded dimension $[n]!$.

It is common—and convenient for calculations—to interpret H_α diagrammatically. In this paragraph we explain this point of view under the simplifying assumption that the underlying quiver is *simply-laced*, i.e. $m_{i,j} + m_{j,i} \leq 1$ for all $i \neq j$. Start with the free graded \mathbb{K} -linear monoidal category H' generated by objects i ($i \in I$) and homogeneous morphisms $x : i \rightarrow i$ of degree 2 and $\tau : ij \rightarrow ji$ of degree $-\alpha_i \cdot \alpha_j$ for all $i, j \in I$. We represent x and τ diagrammatically by:

$$x = \begin{array}{c} i \\ | \\ \bullet \\ | \\ i \end{array} \quad \tau = \begin{array}{c} j & i \\ \diagup & \diagdown \\ i & j \end{array}$$

Composition of morphisms corresponds to vertical concatenation of diagrams (so $a \circ b$ is the diagram a on top of the diagram b), while tensor product is horizontal concatenation (so $a \otimes b$ is a to the left of b). Arbitrary objects are tensor products $i_1 \otimes \dots \otimes i_n$ of the generators $i_1, \dots, i_n \in I$, which we identify with words $\mathbf{i} = i_1 \dots i_n \in \langle I \rangle$. Then, for two words $\mathbf{i}, \mathbf{j} \in \langle I \rangle$, an arbitrary morphism $\mathbf{i} \rightarrow \mathbf{j}$ is a linear combination of diagrams obtained by composing x 's and τ 's horizontally and vertically to obtain braid-like diagrams with strings consistently colored by the letters of the word \mathbf{i} at the bottom and the word \mathbf{j} at the top. In particular, $\text{Hom}_{H'}(\mathbf{i}, \mathbf{j}) = \emptyset$ unless \mathbf{i} and \mathbf{j} both lie in $\langle I \rangle_\alpha$ for some $\alpha \in Q^+$. Then the *quiver*

Hecke category H is the \mathbb{K} -linear monoidal category obtained from H' by imposing the following relations:

$$\begin{aligned}
 \begin{array}{c} \bullet \\ \diagup \\ i \neq j \\ \diagdown \end{array} &= \begin{array}{c} \bullet \\ \diagup \\ i \neq j \\ \diagdown \end{array} \quad \begin{array}{c} \bullet \\ \diagup \\ i \neq j \\ \diagdown \end{array} &= \begin{array}{c} \bullet \\ \diagup \\ i \neq j \\ \diagdown \end{array} \quad \begin{array}{c} \bullet \\ \diagup \\ i \neq j \\ \diagdown \end{array} = \begin{array}{c} \bullet \\ \diagup \\ i \neq j \\ \diagdown \end{array} + \begin{array}{c} | \\ | \\ i \\ i \end{array} \quad \begin{array}{c} \bullet \\ \diagup \\ i \neq j \\ \diagdown \end{array} = \begin{array}{c} \bullet \\ \diagup \\ i \neq j \\ \diagdown \end{array} + \begin{array}{c} | \\ | \\ i \\ i \end{array} \\
 \begin{array}{c} \bullet \\ \diagup \\ i \\ j \\ \diagdown \end{array} &= \begin{cases} 0 & \text{if } i = j \\ \begin{array}{c} | \\ | \\ i \\ j \\ | \\ | \\ i \\ j \end{array} - \begin{array}{c} | \\ | \\ i \\ j \\ | \\ | \\ i \\ j \end{array} & \text{if } i \rightarrow j \\ \begin{array}{c} | \\ | \\ i \\ j \\ | \\ | \\ i \\ j \end{array} - \begin{array}{c} | \\ | \\ i \\ j \\ | \\ | \\ i \\ j \end{array} & \text{if } i \leftarrow j \\ \begin{array}{c} | \\ | \\ i \\ j \end{array} & \text{otherwise,} \end{cases} & \begin{array}{c} \bullet \\ \diagup \\ i \\ j \\ k \\ \diagdown \end{array} - \begin{array}{c} \bullet \\ \diagup \\ i \\ j \\ k \\ \diagdown \end{array} &= \begin{cases} \begin{array}{c} | \\ | \\ i \\ j \\ | \\ | \\ i \\ j \\ k \end{array} & \text{if } i = k \rightarrow j \\ \begin{array}{c} | \\ | \\ i \\ j \\ | \\ | \\ i \\ j \\ k \end{array} & \text{if } i = k \leftarrow j \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

The quiver Hecke algebra $H_\alpha = \bigoplus_{\mathbf{i}, \mathbf{j} \in \langle I \rangle_\alpha} 1_{\mathbf{j}} H_\alpha 1_{\mathbf{i}}$ from before is identified with the vector space $\bigoplus_{\mathbf{i}, \mathbf{j} \in \langle I \rangle_\alpha} \text{Hom}_H(\mathbf{i}, \mathbf{j})$, so that multiplication in H_α corresponds to vertical composition of morphisms in H .

For $\alpha \in Q^+$ of height n again, the relations imply that there is an antiautomorphism $\mathbf{T} : H_\alpha \rightarrow H_\alpha$ defined on generators by

$$\mathbf{T}(1_{\mathbf{i}}) = 1_{\mathbf{i}}, \quad \mathbf{T}(x_k) = x_k, \quad \mathbf{T}(\tau_k) = \tau_k. \quad (2.7)$$

In diagrammatic terms, \mathbf{T} reflects in a horizontal axis.

Basis theorem and center. Suppose in this subsection that $\alpha \in Q^+$ is of height n . In general the braid relations are only approximately true in H_α . So, to write down a basis, we must fix a choice of a distinguished reduced expression $w = t_{i_1} \cdots t_{i_k}$ for each $w \in S_n$, then define $\tau_w := \tau_{i_1} \cdots \tau_{i_k} \in H_\alpha$. Also, for $w \in S_n$ and $\mathbf{i} \in \langle I \rangle_\alpha$, let

$$\deg(w; \mathbf{i}) := - \sum_{\substack{1 \leq j < k \leq n \\ w(j) > w(k)}} \alpha_{i_j} \cdot \alpha_{i_k}. \quad (2.8)$$

The following theorem is proved in a similar way to the special case Theorem 2.3, by constructing a certain *polynomial representation* of the quiver Hecke algebra on the underlying vector space $\text{Pol}_\alpha := \bigoplus_{\mathbf{i} \in \langle I \rangle_\alpha} \text{Pol}_n 1_{\mathbf{i}}$.

Theorem 2.6 (“Basis theorem”). *The monomials*

$$\{x_1^{m_1} \cdots x_n^{m_n} \tau_w 1_{\mathbf{i}} \mid \mathbf{i} \in \langle I \rangle_\alpha, w \in S_n, m_1, \dots, m_n \geq 0\}$$

give a basis for H_α . Hence, for $\mathbf{i}, \mathbf{j} \in \langle I \rangle_\alpha$, we have that

$$\text{Dim } 1_{\mathbf{j}} H_\alpha 1_{\mathbf{i}} = \frac{1}{(1 - q^2)^n} \sum_{\substack{w \in S_n \\ w(\mathbf{i}) = \mathbf{j}}} q^{\deg(w; \mathbf{i})}. \quad (2.9)$$

Note Theorem 2.6 shows in particular that H_α is locally finite dimensional and bounded below. The final basic result in this section is concerned with the center $Z(H_\alpha)$. To formulate it, we pick $\mathbf{i} \in \langle I \rangle_\alpha$ so that $S_{\mathbf{i}} := \text{Stab}_{S_n}(\mathbf{i})$ is a standard parabolic subgroup of S_n , i.e. all equal letters in the word \mathbf{i} appear consecutively. For $j = 1, \dots, n$, let

$$z_j := \sum_{w \in S_n / S_{\mathbf{i}}} x_{w(j)} 1_{w(\mathbf{i})}, \quad (2.10)$$

where $S_n / S_{\mathbf{i}}$ denotes the set of minimal length left coset representatives. In view of Theorem 2.6, these elements generate a free polynomial algebra $\mathbb{K}[z_1, \dots, z_n]$ inside H_α . We let $S_{\mathbf{i}} \leq S_n$ act on $\mathbb{K}[z_1, \dots, z_n]$ by permuting the generators.

Theorem 2.7 (“Center”). *We have that*

$$Z(H_\alpha) = \mathbb{K}[z_1, \dots, z_n]^{S_{\mathbf{i}}}.$$

Hence H_α is free of finite rank as a module over its center; forgetting the grading the rank is $(n!)^2$.

To illustrate how the second statement of the theorem is deduced from the first, consider the special case $I = \{1, 2\}$ and $\alpha = 2\alpha_1 + \alpha_2$, and take $\mathbf{i} = 112$. Then

$$\begin{aligned} z_1 &= x_1 1_{112} + x_1 1_{121} + x_2 1_{211}, \\ z_2 &= x_2 1_{112} + x_3 1_{121} + x_3 1_{211}, \\ z_3 &= x_3 1_{112} + x_2 1_{121} + x_1 1_{211}. \end{aligned}$$

By the first part of the theorem, $Z(H_\alpha)$ is freely generated by the elements $z_1 + z_2$, $z_1 z_2$ and z_3 . The algebra $\mathbb{K}[z_1, z_2, z_3]$ is free with basis $\{1, z_1\}$ as a $Z(H_\alpha)$ -module; the algebra Pol_α embedded in the natural way into H_α is free as a $\mathbb{K}[z_1, z_2, z_3]$ -module with basis $\{1_{112}, 1_{121}, 1_{211}\}$; and finally H_α is a free left Pol_α -module on basis $\{\tau_w \mid w \in S_3\}$. Putting it all together we see that H_α is a free $Z(H_\alpha)$ -module of rank 36.

Notes. Our discussion of nil Hecke algebras follows [31, §2].

Quiver Hecke algebras were introduced by Khovanov and Lauda [18, 19] and independently by Rouquier [30] (in a slightly more general form); consequently they are also often called *Khovanov-Lauda-Rouquier algebras*. We have restricted to symmetric Cartan matrices for simplicity, but note that all of the definitions and results described in this article can be extended to arbitrary symmetrizable Cartan matrices, with the notable exception of Theorem 3.11. The precise normalization of the relations chosen here matches that of [31, §5] and [32], where quiver Hecke algebras for symmetric Cartan matrices are realized geometrically as ext-algebras of a certain direct sum of degree-shifted irreducible perverse sheaves on a quiver variety.

Theorems 2.6 and 2.7 are proved both in [30, Theorem 3.7 and Proposition 3.9] and in [18, Theorem 2.5 and Theorem 2.9].

3. Categorification

Now we describe the main results relating the representation theory of the quiver Hecke algebras H_α (or the quiver Hecke category H) to Lusztig's algebra \mathbf{f} , i.e. half of the quantized enveloping algebra associated to the Kac-Moody algebra \mathfrak{g} .

Rep and Proj. By an H -module we henceforth mean a Q^+ -graded vector space $V = \bigoplus_{\alpha \in Q^+} 1_\alpha V$ such that each $1_\alpha V$ is a *graded* left H_α -module. Occasionally we talk about *ungraded* modules, meaning each $1_\alpha V$ is a left H_α -module without any prescribed grading. In view of Theorem 2.6, each H_α is locally finite dimensional and bounded below, hence irreducible H -modules are automatically finite dimensional. Also Theorem 2.7 implies that there are only finitely many irreducible graded H_α -modules for each $\alpha \in Q^+$ up to isomorphism and degree shift.

Lemma 3.1. *Every irreducible H -module remains irreducible as an ungraded module. If L_1 and L_2 are two irreducible H -modules which are isomorphic as ungraded modules then there exists a unique $m \in \mathbb{Z}$ such that $L_1 \cong q^m L_2$ as graded modules.*

Proof. The first part is [28, Theorem 4.4.4(v)], and the second part follows from [2, Lemma 2.5.3]. \square

We have that $1_\alpha = \bigoplus_{\mathbf{i} \in \langle I \rangle_\alpha} 1_{\mathbf{i}}$ as a sum of mutually orthogonal idempotents, hence any H -module V decomposes as a vector space as $V = \bigoplus_{\mathbf{i} \in \langle I \rangle} 1_{\mathbf{i}} V$. We refer to the subspace $1_{\mathbf{i}} V$ as the *\mathbf{i} -word space* of V . The *character* of a finite dimensional H -module V is defined from

$$\text{Ch } V := \sum_{\mathbf{i} \in \langle I \rangle} (\dim 1_{\mathbf{i}} V) \mathbf{i}, \quad (3.1)$$

which is an element of the free $\mathbb{Z}[q, q^{-1}]$ -module $\mathbb{Z}[q, q^{-1}]\langle I \rangle$ on basis $\langle I \rangle$. Forgetting the grading, we also have the *ungraded character* $\text{ch } V := \sum_{\mathbf{i} \in \langle I \rangle} (\dim 1_{\mathbf{i}} V) \mathbf{i}$, which is the specialization of $\text{Ch } V$ at $q = 1$.

Lemma 3.2. *Fix a total order $<$ on I . For a word $\mathbf{i} \in \langle I \rangle$ of length n define*

$$p(\mathbf{i}) := \sum_{\substack{1 \leq j < k \leq n \\ i_j < i_k}} \alpha_{i_j} \cdot \alpha_{i_k} \pmod{2}.$$

Then every H -module V decomposes as a direct sum of modules as $V = V^{\bar{0}} \oplus V^{\bar{1}}$ where

$$V^q := \bigoplus_{\substack{\mathbf{i} \in \langle I \rangle, n \in \mathbb{Z} \\ n \equiv p(\mathbf{i}) + q \pmod{2}}} 1_{\mathbf{i}} V_n.$$

Proof. We just need to check for $\alpha \in Q^+$ that $1_\alpha V^{\bar{0}}$ and $1_\alpha V^{\bar{1}}$ are stable under the action of the generators of H_α . This is clear for the generators $1_{\mathbf{i}}$ and x_j since these are even and preserve the word spaces. For τ_k , note that it maps $1_{\mathbf{i}} V_n$ into $1_{t_k(\mathbf{i})} V_{n - \alpha_{i_k} \cdot \alpha_{i_{k+1}}}$, and we have that $p(t_k(\mathbf{i})) \equiv p(\mathbf{i}) - \alpha_{i_k} \cdot \alpha_{i_{k+1}} \pmod{2}$. \square

We are going to focus now on the categories

$$\text{Rep}(H) = \bigoplus_{\alpha \in Q^+} \text{Rep}(H_\alpha), \quad \text{Proj}(H) = \bigoplus_{\alpha \in Q^+} \text{Proj}(H_\alpha)$$

of finite dimensional H -modules and finitely generated projective H -modules, respectively. Morphisms in both categories are module homomorphisms that are homogeneous of degree 0. In particular, this ensures that $\text{Rep}(H)$ is an abelian category. We continue to write $\text{Hom}_H(V, W)$ for $\bigoplus_{n \in \mathbb{Z}} \text{Hom}_H(V, W)_n$, where $\text{Hom}_H(V, W)_n$ means homomorphisms that are homogeneous of degree n . In other words, we are viewing $\text{Rep}(H)$ and $\text{Proj}(H)$ as *graded categories* equipped with the distinguished *degree shift automorphism* $V \mapsto qV$, and $\text{Hom}_H(V, W)$ is the graded vector space $\bigoplus_{n \in \mathbb{Z}} \text{hom}(q^n V, W)$ defined via the internal hom.

The following theorem is fundamental. Its proof is essentially the same as the proof of the analogous statement for the affine Hecke algebra H_n from the introduction, which is classical. The argument uses the Shuffle Lemma (formulated as Corollary 3.7 below) and the properties of the nil Hecke algebra proved already.

Theorem 3.3 (“Linear independence of characters”). *For $\alpha \in Q^+$, suppose that L_1, \dots, L_r are representatives for the irreducible graded left H_α -modules up to isomorphism and degree shift. Their ungraded characters $\text{ch} L_1, \dots, \text{ch} L_r \in \mathbb{Z}\langle I \rangle$ are \mathbb{Z} -linearly independent.*

Let $[\text{Rep}(H)]$ denote the Grothendieck group of the abelian category $\text{Rep}(H)$ and $[\text{Proj}(H)]$ be the split Grothendieck group of the additive category $\text{Proj}(H)$. The isomorphism classes of irreducible H -modules give a basis for $[\text{Rep}(H)]$, and the isomorphism classes of projective indecomposable H -modules give a basis for $[\text{Proj}(H)]$, both as free \mathbb{Z} -modules. We often view $[\text{Rep}(H)]$ and $[\text{Proj}(H)]$ as $\mathbb{Z}[q, q^{-1}]$ -modules, with action of q induced by the degree shift automorphism. The character map from (3.1) induces a $\mathbb{Z}[q, q^{-1}]$ -linear map

$$\text{Ch} : [\text{Rep}(H)] \rightarrow \mathbb{Z}[q, q^{-1}]\langle I \rangle.$$

Corollary 3.4. *The map $\text{Ch} : [\text{Rep}(H)] \rightarrow \mathbb{Z}[q, q^{-1}]\langle I \rangle$ is injective.*

Proof. Let L_1, \dots, L_r be as in Theorem 3.3, so that $[L_1], \dots, [L_r]$ give a basis for $[\text{Rep}(H)]$ as a free $\mathbb{Z}[q, q^{-1}]$ -module. We need to show that their characters $\text{Ch} L_1, \dots, \text{Ch} L_r$ are $\mathbb{Z}[q, q^{-1}]$ -linearly independent. Take a non-trivial linear relation $\sum_{i=1}^r f_i(q) \text{Ch} L_i = 0$ for $f_i(q) \in \mathbb{Z}[q, q^{-1}]$. Dividing by $q-1$ if necessary we may assume that $f_i(1) \neq 0$ for some i . Then we specialize at $q=1$ to deduce that $\sum_{i=1}^r f_i(1) \text{ch} L_i = 0$, contradicting Theorem 3.3. \square

There are dualities \circledast and $\#$ on $\text{Rep}(H)$ and $\text{Proj}(H)$, respectively, defined by $V^\circledast := \text{Hom}_{\mathbb{K}}(V, \mathbb{K})$ and $P^\# := \text{Hom}_H(P, H)$, with the left action of H arising via the antiautomorphism T . These dualities induce involutions of the Grothendieck groups $[\text{Rep}(H)]$ and $[\text{Proj}(H)]$ which are antilinear with respect to the bar involution $- : \mathbb{Z}[q, q^{-1}] \rightarrow \mathbb{Z}[q, q^{-1}]$, $q \mapsto q^{-1}$. There is also a non-degenerate bilinear pairing $(\cdot, \cdot) : [\text{Proj}(H)] \times [\text{Rep}(H)] \rightarrow \mathbb{Z}[q, q^{-1}]$ defined from

$$([P], [V]) := \text{Dim} \text{Hom}_H(P^\#, V) = \overline{\text{Dim} \text{Hom}_H(P, V^\circledast)}. \quad (3.2)$$

Lemma 3.5. *Any irreducible H -module L can be shifted uniquely in degree so that it becomes \circledast -self-dual. In that case its projective cover P is $\#$ -self-dual.*

Proof. Let L be an irreducible H -module. It is immediate from the definition of \circledast that the word spaces of L and L^\circledast have the same dimensions (although they might not have the same graded dimensions). In other words, we have that $\text{ch } L = \text{ch}(L^\circledast)$. Combined with Theorem 3.3 and Lemma 3.1, this means that there is a unique $m \in \mathbb{Z}$ such that $L^\circledast \cong q^m L$. Now pick i such that $1_i L \neq 0$. By Lemma 3.2 we have that

$$\text{Dim } 1_i L = a_0 q^p + a_1 q^{p+2} + \cdots + a_k q^{p+2k}$$

for some $p \in \mathbb{Z}$, $k \geq 0$ and $a_0, \dots, a_k \in \mathbb{N}$ with $a_0, a_k \neq 0$. Hence

$$\text{Dim } 1_i L^\circledast = q^m (a_0 q^p + \cdots + a_k q^{p+2k}) = a_k q^{-p-2k} + \cdots + a_0 q^{-p}.$$

We deduce that $m + p = -p - 2k$, hence m is even. Then $(q^{m/2} L)^\circledast \cong q^{-m/2} L^\circledast \cong q^{m/2} L$ and we have proved the first part. For the second part, suppose that $L \cong L^\circledast$ and that P is the projective cover of L . We need to show that the projective indecomposable module $P^\#$ also covers L . This follows because $\dim \text{Hom}_H(P^\#, L)_0 = \dim \text{Hom}_H(P, L^\circledast)_0 = \dim \text{Hom}_H(P, L)_0 \neq 0$, using (3.2). \square

Lemma 3.5 implies that the sets

$$\mathbf{B} := \{[P] \mid \text{for all } \# \text{-self-dual projective indecomposable } H\text{-modules } P\}, \quad (3.3)$$

$$\mathbf{B}^* := \{[L] \mid \text{for all } \circledast \text{-self-dual irreducible } H\text{-modules } L\} \quad (3.4)$$

give bases for $\text{Proj}(H)$ and $\text{Rep}(H)$, respectively, as free $\mathbb{Z}[q, q^{-1}]$ -modules. Moreover these two bases are dual to each other with respect to the pairing (\cdot, \cdot) .

Induction and restriction. For $\beta, \gamma \in Q^+$, tensor product (horizontal composition) in the quiver Hecke category defines a non-unital algebra embedding $H_\beta \otimes H_\gamma \hookrightarrow H_{\beta+\gamma}$. We denote the image of the identity $1_\beta \otimes 1_\gamma \in H_\beta \otimes H_\gamma$ by $1_{\beta, \gamma} \in H_{\beta+\gamma}$. Then for a graded left $H_{\beta+\gamma}$ -module U and a graded left $H_\beta \otimes H_\gamma$ -module V , we set

$$\text{Res}_{\beta, \gamma}^{\beta+\gamma} U := 1_{\beta, \gamma} U, \quad \text{Ind}_{\beta, \gamma}^{\beta+\gamma} V := H_{\beta+\gamma} 1_{\beta, \gamma} \otimes_{H_\beta \otimes H_\gamma} V,$$

which are naturally graded left $H_\beta \otimes H_\gamma$ - and $H_{\beta+\gamma}$ -modules, respectively. Both are exact functors; for induction this is a consequence of the basis theorem (Theorem 2.6). The following Mackey-type theorem is one of the main reasons quiver Hecke algebras are so amenable to purely algebraic techniques.

Theorem 3.6 (“Mackey filtration”). *Suppose $\beta, \gamma, \beta', \gamma' \in Q^+$ are of heights m, n, m', n' , respectively, such that $\beta + \gamma = \beta' + \gamma'$. Setting $k := \min(m, n, m', n')$, let $\{1 = w_0 < \cdots < w_k\}$ be the set of minimal length $S_{m'} \times S_{n'} \setminus S_{m+n} / S_m \times S_n$ -double coset representatives ordered via the Bruhat order. For any graded left $H_\beta \otimes H_\gamma$ -module V , there is a filtration*

$$0 = V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = \text{Res}_{\beta', \gamma'}^{\beta'+\gamma'} \circ \text{Ind}_{\beta, \gamma}^{\beta+\gamma}(V)$$

defined by $V_j := \sum_{i=0}^j \sum_{w \in (S_{m'} \times S_{n'}) w_i (S_m \times S_n)} 1_{\beta', \gamma'} \tau_w 1_{\beta, \gamma} \otimes V$. Moreover there is a unique isomorphism of graded $H_{\beta'} \otimes H_{\gamma'}$ -modules

$$V_j/V_{j-1} \xrightarrow{\sim} \bigoplus_{\beta_1, \beta_2, \gamma_1, \gamma_2} q^{-\beta_2 \cdot \gamma_1} \text{Ind}_{\beta_1, \gamma_1, \beta_2, \gamma_2}^{\beta', \gamma'} \circ I^* \circ \text{Res}_{\beta_1, \beta_2, \gamma_1, \gamma_2}^{\beta, \gamma} (V),$$

$$1_{\beta', \gamma'} \tau_{w_j} 1_{\beta, \gamma} \otimes v + V_{j-1} \mapsto \sum_{\beta_1, \beta_2, \gamma_1, \gamma_2} 1_{\beta_1, \gamma_1, \beta_2, \gamma_2} \otimes 1_{\beta_1, \beta_2, \gamma_1, \gamma_2} v,$$

where $I : H_{\beta_1} \otimes H_{\gamma_1} \otimes H_{\beta_2} \otimes H_{\gamma_2} \xrightarrow{\sim} H_{\beta_1} \otimes H_{\beta_2} \otimes H_{\gamma_1} \otimes H_{\gamma_2}$ is the obvious isomorphism, and the sums are taken over all $\beta_1, \beta_2, \gamma_1, \gamma_2 \in Q^+$ such that $\beta_1 + \beta_2 = \beta, \gamma_1 + \gamma_2 = \gamma, \beta_1 + \gamma_1 = \beta', \beta_2 + \gamma_2 = \gamma'$ and $\min(\text{ht}(\beta_2), \text{ht}(\gamma_1)) = j$.

Proof. Think about the picture

$$w_j = \begin{array}{c} \beta' \quad \gamma' \\ \overbrace{\beta_1 \quad \gamma_1}^{} \quad \overbrace{\beta_2 \quad \gamma_2}^{} \\ \beta \quad \gamma \\ \beta_1 \quad \beta_2 \quad \gamma_1 \quad \gamma_2 \end{array}$$

□

If X is a graded left H_{β} -module and Y is a graded left H_{γ} -module, hence their outer tensor product $X \boxtimes Y$ is a graded left $H_{\beta} \otimes H_{\gamma}$ -module, we set

$$X \circ Y := \text{Ind}_{\beta, \gamma}^{\beta + \gamma} (X \boxtimes Y).$$

This is finite dimensional if both X and Y are finite dimensional (by the basis theorem), and it is projective if both X and Y are projective (indeed we obviously have that $H1_i \circ H1_j = H1_{ij}$). Hence \circ defines a tensor product operation on both $\text{Rep}(H)$ and $\text{Proj}(H)$, making these into monoidal categories. So the Grothendieck groups $[\text{Rep}(H)]$ and $[\text{Proj}(H)]$ become $\mathbb{Z}[q, q^{-1}]$ -algebras. Also introduce a bilinear multiplication \circ on $\mathbb{Z}[q, q^{-1}] \langle I \rangle$, which we call the *shuffle product*, by declaring that for words \mathbf{i} and \mathbf{j} of lengths m and n , respectively, that

$$\mathbf{i} \circ \mathbf{j} := \sum_{\substack{w \in S_{m+n} \\ w(1) < \dots < w(m) \\ w(m+1) < \dots < w(m+n)}} q^{\deg(w; \mathbf{ij})} w(\mathbf{ij}), \quad (3.5)$$

recalling (2.8). This makes $\mathbb{Z}[q, q^{-1}] \langle I \rangle$ into an associative $\mathbb{Z}[q, q^{-1}]$ -algebra, which we call the *quantum shuffle algebra*. The first important consequence of the Mackey theorem (iterated!) is as follows.

Corollary 3.7 (“Shuffle Lemma”). *For finite dimensional H -modules X and Y , we have that*

$$\mathrm{Ch}(X \circ Y) = (\mathrm{Ch} X) \circ (\mathrm{Ch} Y).$$

Hence $\mathrm{Ch} : [\mathrm{Rep}(H)] \hookrightarrow \mathbb{Z}[q, q^{-1}]\langle I \rangle$ is an injective algebra homomorphism.

The restriction functor $\mathrm{Res}_{\beta, \gamma}^{\beta+\gamma}$ sends finite dimensional modules to finite dimensional modules (obviously) and projectives to projectives (by the basis theorem). Summing over all β, γ , we get induced $\mathbb{Z}[q, q^{-1}]$ -module homomorphisms $[\mathrm{Rep}(H)] \rightarrow [\mathrm{Rep}(H)] \otimes_{\mathbb{Z}[q, q^{-1}]} [\mathrm{Rep}(H)]$ and $[\mathrm{Proj}(H)] \rightarrow [\mathrm{Proj}(H)] \otimes_{\mathbb{Z}[q, q^{-1}]} [\mathrm{Proj}(H)]$ making our Grothendieck groups into coalgebras. The Mackey theorem also implies the following.

Corollary 3.8. *The Grothendieck group $[\mathrm{Rep}(H)]$ is a twisted bialgebra (i.e. comultiplication is an algebra homomorphism) with respect to the multiplication on $[\mathrm{Rep}(H)] \otimes_{\mathbb{Z}[q, q^{-1}]} [\mathrm{Rep}(H)]$ defined by*

$$(a \otimes b)(c \otimes d) := q^{-\beta \cdot \gamma} ac \otimes bd \quad (3.6)$$

for $a \in [\mathrm{Rep}(H_\alpha)]$, $b \in [\mathrm{Rep}(H_\beta)]$, $c \in [\mathrm{Rep}(H_\gamma)]$ and $d \in [\mathrm{Rep}(H_\delta)]$. Similarly $[\mathrm{Proj}(H)]$ is a twisted bialgebra in the same sense.

Note finally if we extend the pairing between $[\mathrm{Proj}(H)]$ and $[\mathrm{Rep}(H)]$ diagonally to a pairing between $[\mathrm{Proj}(H)] \otimes_{\mathbb{Z}[q, q^{-1}]} [\mathrm{Proj}(H)]$ and $[\mathrm{Rep}(H)] \otimes_{\mathbb{Z}[q, q^{-1}]} [\mathrm{Rep}(H)]$, i.e. we set $(a \otimes b, c \otimes d) := (a, c)(b, d)$, then the multiplication on $[\mathrm{Rep}(H)]$ is dual to the comultiplication on $[\mathrm{Proj}(H)]$, and vice versa. This follows from the definition (3.2) together with Frobenius reciprocity. In other words the Q^+ -graded twisted bialgebras $[\mathrm{Rep}(H)]$ and $[\mathrm{Proj}(H)]$ are graded dual to each other.

The categorification theorem. We now connect $[\mathrm{Rep}(H)]$ and $[\mathrm{Proj}(H)]$ to Lusztig’s algebra \mathbf{f} , that is, the $\mathbb{Q}(q)$ -algebra on generators θ_i ($i \in I$) subject to the *quantum Serre relations*

$$\sum_{r+s=1-\alpha_i \cdot \alpha_j} (-1)^r \theta_i^{(r)} \theta_j \theta_i^{(s)} = 0$$

for all $i, j \in I$ and $r \geq 1$, where $\theta_i^{(r)}$ denotes the *divided power* $\theta_i^r/[r]!$. There is a Q^+ -grading $\mathbf{f} = \bigoplus_{\alpha \in Q^+} \mathbf{f}_\alpha$ defined so that θ_i is in degree α_i . We use this to make $\mathbf{f} \otimes \mathbf{f}$ into an algebra with twisted multiplication defined in the same way as (3.6). Then there is a unique algebra homomorphism

$$r : \mathbf{f} \rightarrow \mathbf{f} \otimes \mathbf{f}, \quad \theta_i \mapsto \theta_i \otimes 1 + 1 \otimes \theta_i \quad (3.7)$$

making \mathbf{f} into a twisted bialgebra. We also need Lusztig’s $\mathbb{Z}[q, q^{-1}]$ -form $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]}$ for \mathbf{f} , which is the $\mathbb{Z}[q, q^{-1}]$ -subalgebra of \mathbf{f} generated by all $\theta_i^{(r)}$. We always identify \mathbf{f} with $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbf{f}_{\mathbb{Z}[q, q^{-1}]}$. The first important categorification theorem is as follows.

Theorem 3.9 (Khovanov-Lauda). *There is an isomorphism of twisted bialgebras*

$$\gamma : \mathbf{f}_{\mathbb{Z}[q, q^{-1}]} \xrightarrow{\sim} [\text{Proj}(H)]$$

such that $\theta_i \mapsto [H_{\alpha_i}]$ for each $i \in I$.

In the rest of the subsection we are going to explain the proof of this, since it is instructive, elementary, and reveals many further connections between \mathbf{f} and H . To start with we extend scalars and show that there is a $\mathbb{Q}(q)$ -algebra homomorphism

$$\hat{\gamma} : \mathbf{f} \rightarrow \mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Proj}(H)], \quad \theta_i \mapsto [H_{\alpha_i}].$$

To prove this we must show that the Serre relations hold in the right hand algebra. For $\mathbf{i} = i_1 \cdots i_n \in \langle I \rangle$ let $\theta_{\mathbf{i}} := \theta_{i_1} \cdots \theta_{i_n}$. Note that the homomorphism γ (if it exists) should send $\theta_{\mathbf{i}}$ to $[H_{\mathbf{i}}]$. Hence, recalling the definition (2.6) and using Corollary 2.4, it should send $\theta_{\mathbf{i}}^{(n)}$ to $[P(i^n)]$. Thus we need to show that

$$\sum_{r+s=1-\alpha_i \cdot \alpha_j} (-1)^r [P(i^r) \circ P(j) \circ P(i^s)] = 0$$

in $[\text{Proj}(H)]$. This follows from the following stronger statement, which gives a categorification of the Serre relations.

Lemma 3.10. *Take $i, j \in I$ with $i \neq j$ and let $n := 1 - \alpha_i \cdot \alpha_j$. Let $\alpha := n\alpha_i + \alpha_j$. For $r, s \geq 0$ with $r + s = n$, let $e_{r,s} \in H_{\alpha}$ denote the image of $e_r \otimes 1_j \otimes e_s$ under the embedding $H_{r\alpha_i} \otimes H_{\alpha_j} \otimes H_{s\alpha_i} \hookrightarrow H_{\alpha}$, so that $P(i^r) \circ P(j) \circ P(i^s) = q^{\frac{1}{2}r(r-1) + \frac{1}{2}s(s-1)} H_{\alpha} e_{r,s}$. Then there is an exact sequence*

$$\begin{aligned} 0 \rightarrow q^{\frac{1}{2}n(n-1)} H_{\alpha} e_{0,n} &\rightarrow \cdots \\ &\rightarrow q^{\frac{1}{2}r(r-1) + \frac{1}{2}s(s-1)} H_{\alpha} e_{r,s} \xrightarrow{d_{r,s}} q^{\frac{1}{2}(r+1)r + \frac{1}{2}(s-1)(s-2)} H_{\alpha} e_{r+1,s-1} \rightarrow \\ &\cdots \rightarrow q^{\frac{1}{2}n(n-1)} H_{\alpha} e_{n,0} \rightarrow 0. \end{aligned}$$

The homomorphism $d_{r,s}$ is given by right multiplication by

$$\tau_{r,s} := \begin{array}{c} \text{Diagram showing two sets of vertical lines labeled } i \text{ and } j. \\ \text{The top set has } r \text{ lines labeled } i \text{ and } 1 \text{ line labeled } j. \\ \text{The bottom set has } s \text{ lines labeled } i \text{ and } 1 \text{ line labeled } j. \\ \text{Crossed lines: } i_1 \text{ (top) } \leftrightarrow j_1 \text{ (bottom)}, i_2 \text{ (top) } \leftrightarrow j_2 \text{ (bottom)}, \dots, i_r \text{ (top) } \leftrightarrow j_r \text{ (bottom)}. \\ \text{Underneath the lines: } r+1 \text{ (under the first } i\text{'s), } s-1 \text{ (under the last } i\text{'s).} \end{array}$$

Proof. It is a complex because $e_{r,s} \tau_{r,s} \tau_{r+1,s-1} = 0$. To see this observe that the product $\tau_{r,s} \tau_{r+1,s-1}$ can be rewritten as $\tau_{r+2} \pi$ for some π , and $e_{r,s} \tau_{r+2} = 0$ by the definition (2.4). It is exact because it is homotopy equivalent to 0. To see this

let $d'_{r,s} : q^{\frac{1}{2}(r+1)r + \frac{1}{2}(s-1)(s-2)} H_\alpha e_{r+1,s-1} \rightarrow q^{\frac{1}{2}r(r-1) + \frac{1}{2}s(s-1)} H_\alpha e_{r,s}$ be defined by right multiplication by

$$\tau'_{r,s} := (-1)^{r+m_{j,i}} \begin{array}{c} \text{Diagram showing a permutation of } r+s \text{ elements.} \\ \text{Top row: } i, i, i, i, i, j, i, i, i, i \\ \text{Bottom row: } i, i, i, i, j, i, i, i, i, i \\ \text{Curly braces: } r \text{ (under the first 5 elements), } s \text{ (under the last 5 elements).} \\ \text{Crossed lines: } \text{A line from the } j \text{ in the top row to the } i \text{ in the bottom row, and a line from the } i \text{ in the top row to the } j \text{ in the bottom row.} \end{array}$$

Then check miraculously that $d'_{r,s} \circ d_{r,s} + d_{r-1,s+1} \circ d'_{r-1,s+1} = \text{id}$. \square

Hence the homomorphism $\hat{\gamma}$ is well-defined. Moreover both \mathbf{f} and $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} [\text{Proj}(H)]$ are twisted bialgebras in the same sense, $\hat{\gamma}$ respects the Q^+ -gradings, and it is easy to see that the analogue of (3.7) also holds in $[\text{Proj}(H)]$. So $\hat{\gamma}$ is actually a homomorphism of twisted bialgebras. We've already observed that it sends $\theta_i^{(r)}$ to $[P(i^n)] \in [\text{Proj}(H)] \subset \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} [\text{Proj}(H)]$. So $\hat{\gamma}$ restricts to a well-defined homomorphism γ as in the statement of Theorem 3.9.

It remains to show that γ is an isomorphism. To establish its surjectivity, we dualize and exploit Corollary 3.4. In more detail, Lusztig showed that \mathbf{f} possesses a unique non-degenerate symmetric bilinear form (\cdot, \cdot) such that

$$(1, 1) = 1, \quad (\theta_i, \theta_j) = \frac{\delta_{i,j}}{1 - q^2}, \quad (ab, c) = (a \otimes b, r(c))$$

for all $i, j \in I$ and $a, b, c \in \mathbf{f}$; on the right hand side of the last equation (\cdot, \cdot) is the product form $(a \otimes b, c \otimes d) := (a, c)(b, d)$ on $\mathbf{f} \otimes \mathbf{f}$. Let $\mathbf{f}_{\mathbb{Z}[q,q^{-1}]}^*$ be the dual of $\mathbf{f}_{\mathbb{Z}[q,q^{-1}]}$ with respect to the form (\cdot, \cdot) , i.e.

$$\mathbf{f}_{\mathbb{Z}[q,q^{-1}]}^* := \{y \in \mathbf{f} \mid (x, y) \in \mathbb{Z}[q, q^{-1}] \text{ for all } x \in \mathbf{f}_{\mathbb{Z}[q,q^{-1}]}\}.$$

It is another $\mathbb{Z}[q, q^{-1}]$ -form for \mathbf{f} , that is, it is a twisted bialgebra such that $\mathbf{f} = \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbf{f}_{\mathbb{Z}[q,q^{-1}]}^*$. Taking the dual map to γ with respect to Lusztig's pairing between $\mathbf{f}_{\mathbb{Z}[q,q^{-1}]}$ and $\mathbf{f}_{\mathbb{Z}[q,q^{-1}]}^*$ and the pairing (3.2) between $[\text{Proj}(H)]$ and $[\text{Rep}(H)]$ gives a twisted bialgebra homomorphism

$$\gamma^* : [\text{Rep}(H)] \rightarrow \mathbf{f}_{\mathbb{Z}[q,q^{-1}]}^*.$$

For $x \in \mathbf{f}_{\mathbb{Z}[q,q^{-1}]}^*$, we let $\chi(x) := \sum_{i \in \langle I \rangle} (\theta_i, x) i$, thus defining an injective map $\chi : \mathbf{f}_{\mathbb{Z}[q,q^{-1}]}^* \hookrightarrow \mathbb{Z}[q, q^{-1}] \langle I \rangle$. Then we claim that the following diagram commutes:

$$\begin{array}{ccc} [\text{Rep}(H)] & \xrightarrow{\gamma^*} & \mathbf{f}_{\mathbb{Z}[q,q^{-1}]}^* \\ \text{Ch} \searrow & & \swarrow \chi \\ & \mathbb{Z}[q, q^{-1}] \langle I \rangle & \end{array}$$

To see this, just observe for a finite dimensional H -module V that

$$(\theta_i, \gamma^*[V]) = (\gamma(\theta_i), [V]) = (H1_i, [V]) = \text{Dim Hom}_H(H1_i, V) = \text{Dim } 1_i V, \quad (3.8)$$

which is indeed the coefficient of 1_i in $\text{Ch } V$. The map Ch is injective by Corollary 3.4, hence γ^* is injective, and γ is surjective as required.

Finally we must show that γ is injective. To see this we lift the pairing between $[\text{Proj}(H)]$ and $[\text{Rep}(H)]$ to define a bilinear form on $[\text{Proj}(H)]$ such that

$$([P], [Q]) := \text{Dim Hom}_H(P^\#, Q).$$

At first sight this takes values in the ring of formal Laurent series in q . However, the surjectivity established in the previous paragraph implies that $[\text{Proj}(H)]$ is spanned by the classes $\{[H1_i] \mid i \in \langle I \rangle\}$, and $([H1_j], [H1_i]) = \text{Dim } 1_j H1_i$, which lies in $\mathbb{Q}(q)$ by (2.9). Thus our form takes values in $\mathbb{Q}(q)$ and, extending scalars, we get induced a symmetric $\mathbb{Q}(q)$ -bilinear form (\cdot, \cdot) on $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Proj}(H)]$. It remains to observe from the definition of Lusztig's form above that

$$(\theta_j, \theta_i) = \frac{1}{(1 - q^2)^n} \sum_{\substack{w \in S_n \\ w(i) = j}} q^{\deg(w; i)} \quad (3.9)$$

for $i, j \in \langle I \rangle$ with i of length n . This agrees with the right hand side of (2.9), hence $(\hat{\gamma}(x), \hat{\gamma}(y)) = (x, y)$ for all $x, y \in \mathbf{f}$. Now if $\hat{\gamma}(x) = 0$ for some $x \in \mathbf{f}$ we deduce from this that $(x, y) = 0$ for all $y \in \mathbf{f}$, hence $x = 0$ by the non-degeneracy of Lusztig's form. This shows that $\hat{\gamma}$, hence γ , is injective. This completes the proof of Theorem 3.9.

In the above argument, we have shown not only that $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]} \cong [\text{Proj}(H)]$ but also that $[\text{Rep}(H)] \cong \mathbf{f}_{\mathbb{Z}[q, q^{-1}]}$ (both as twisted bialgebras). We have also identified Lusztig's form with our representation-theoretically defined form (3.2). There is one other useful identification to be made at this point. Let $\mathbf{b} : \mathbf{f} \rightarrow \mathbf{f}$ be the anti-linear algebra automorphism such that $\mathbf{b}(\theta_i) = \theta_i$ for all $i \in I$. Also let $\mathbf{b}^* : \mathbf{f} \rightarrow \mathbf{f}$ be the adjoint anti-linear map to \mathbf{b} with respect to Lusztig's form, so \mathbf{b}^* is defined from $(x, \mathbf{b}^*(y)) = \overline{(\mathbf{b}(x), y)}$ for any $x, y \in \mathbf{f}$. The maps \mathbf{b} and \mathbf{b}^* preserve $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]}$ and $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]}$, respectively. The map \mathbf{b}^* is not an algebra homomorphism; instead it has the property

$$\mathbf{b}^*(xy) = q^{\beta \cdot \gamma} \mathbf{b}^*(y) \mathbf{b}^*(x) \quad (3.10)$$

for x of weight β and y of weight γ . It is obvious for any $i \in \langle I \rangle$ that $\mathbf{b}(\theta_i) = \theta_i$ and $H1_i^\# = H1_i$, which is all that is needed to prove that the isomorphism γ intertwines \mathbf{b} with the anti-linear involution on $[\text{Proj}(H)]$ induced by the duality $\#$. Because \circledast is adjoint to $\#$ thanks to (3.2), we deduce that γ^* intertwines \mathbf{b}^* with the anti-linear involution on $[\text{Rep}(H)]$ induced by the duality \circledast .

For the remainder of the article we will simply *identify* $[\text{Proj}(H)]$ with $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]}$ and $[\text{Rep}(H)]$ with $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]}$ via the maps γ and γ^* . In particular the bases \mathbf{B} and \mathbf{B}^* from (3.3)–(3.4) give bases for $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]}$ and $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]}$, respectively. Here is the next remarkable result.

Theorem 3.11 (Rouquier, Varagnolo-Vasserot). *Assume \mathbb{K} is of characteristic 0. Then \mathbf{B} coincides with the Lusztig-Kashiwara canonical basis for \mathbf{f} , and \mathbf{B}^* is the dual canonical basis.*

Some examples. Here we give a few rather special examples which illustrate the algebraic techniques (characters, shuffle products, etc...) that we have developed so far. First suppose that the graph underlying the quiver is the Dynkin diagram 1—2 of type A_2 , and consider H_α for the highest root $\alpha = \alpha_1 + \alpha_2$. By Frobenius reciprocity, any irreducible graded H_α -module must appear in the head of one of $L(1) \circ L(2)$ or $L(2) \circ L(1)$. By the Shuffle Lemma, we have that

$$\mathrm{Ch}(L(1) \circ L(2)) = 12 + q21, \quad \mathrm{Ch}(L(2) \circ L(1)) = 21 + q12.$$

Because of Corollary 3.5 this means that there can only be two irreducible graded H_α -modules up to isomorphism and degree shift, namely, the one-dimensional modules $L(12)$ and $L(21)$ with characters 12 and 21, respectively. So in this case, H_α is already a *basic algebra*. In fact it is easy to see directly that H_α is isomorphic to $A \otimes \mathbb{K}[x]$, where A is the path algebra of the quiver

$$\bullet \xrightleftharpoons[\tau]{\tau} \bullet \tag{3.11}$$

graded by path length and x is of degree 2 (corresponding to the central element $x_1 1_{12} + x_2 1_{21} \in H_\alpha$). We deduce from this that H_α has global dimension 2.

We pause briefly to discuss *homogeneous representations*. So return for a moment to a general quiver Hecke algebra. Let \sim be the equivalence relation on $\langle I \rangle$ generated by permuting adjacent pairs of letters ij in a word whenever $\alpha_i \cdot \alpha_j = 0$. Call a word $\mathbf{i} \in \langle I \rangle$ a *homogeneous word* if every $\mathbf{j} = j_1 \cdots j_n \sim \mathbf{i}$ satisfies the following conditions:

- ▷ $j_k \neq j_{k+1}$ for each $k = 1, \dots, n-1$;
- ▷ if $j_k = j_{k+2}$ for some $k = 1, \dots, n-2$ then $\alpha_{j_k} \cdot \alpha_{j_{k+1}} \neq -1$.

In that case, by the relations, there is a well-defined H -module $L(\mathbf{i})$ concentrated in degree zero with basis $\{v_{\mathbf{j}} \mid \mathbf{j} \sim \mathbf{i}\}$ such that

- ▷ $1_{\mathbf{k}} v_{\mathbf{j}} = \delta_{\mathbf{j}, \mathbf{k}} v_{\mathbf{j}}$;
- ▷ $x_k v_{\mathbf{j}} = 0$;
- ▷ $\tau_k v_{\mathbf{j}} = v_{t_k(\mathbf{j})}$ if $\alpha_{j_k} \cdot \alpha_{j_{k+1}} = 0$, $\tau_k v_{\mathbf{j}} = 0$ otherwise.

It is obvious that $L(\mathbf{i})$ is irreducible. Moreover for two homogeneous words \mathbf{i}, \mathbf{j} , we have that $L(\mathbf{i}) \cong L(\mathbf{j})$ if and only if $\mathbf{i} \sim \mathbf{j}$.

Theorem 3.12 (Kleshchev-Ram). *Let Ω be a set of representatives for the \sim -equivalence classes of homogeneous words $\mathbf{i} \in \langle I \rangle$. Up to isomorphism, the modules $\{L(\mathbf{i}) \mid \mathbf{i} \in \Omega\}$ give all irreducible H -modules that are concentrated in degree zero.*

Proof. Let L be an irreducible H -module that is concentrated in degree zero. Let \mathbf{i} be a word of L , i.e. a word such that $1_{\mathbf{i}}L \neq 0$. We claim first that \mathbf{i} is homogeneous. To see this, note for $\mathbf{j} \sim \mathbf{i}$ that $\dim 1_{\mathbf{j}}L = \dim 1_{\mathbf{i}}L$ since $\tau_k^2 1_{\mathbf{i}} = 1_{\mathbf{i}}$ whenever $\alpha_{i_k} \cdot \alpha_{i_{k+1}} = 0$. Hence \mathbf{j} is also a word of L . If $j_k = j_{k+1}$ for some k we get a contradiction to the relation $(\tau_k x_{k+1} - x_k \tau_k)1_{\mathbf{j}} = 1_{\mathbf{j}}$, since the left hand side has to act on L as zero by degree considerations. Similarly if $j_k = j_{k+2}$ and $\alpha_{j_k} \cdot \alpha_{j_{k+1}} = -1$ for some k we get a contradiction to the relation $(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k)1_{\mathbf{j}} = \pm 1_{\mathbf{j}}$. To complete the proof of the theorem, it remains to observe that

$$\text{Ch } L = \sum_{\mathbf{i} \in \Omega} (\dim 1_{\mathbf{i}}L) \text{Ch } L(\mathbf{i})$$

Hence $L \cong L(\mathbf{i})$ for some $\mathbf{i} \in \Omega$ by Corollary 3.4. \square

Now back to examples. We next suppose that the graph underlying our quiver is the A_3 Dynkin diagram 1—2—3, and take $\alpha = \alpha_1 + \alpha_2 + \alpha_3$. Because α is multiplicity-free, all words $\mathbf{i} \in \langle I \rangle_\alpha$ are homogeneous, hence all irreducible graded H_α -modules are homogeneous representations. Here are all of the skew-hooks with three boxes:

$$\begin{array}{c|c|c|c} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{3} \\ \hline 1 & & & 2 \\ & & & 1 \end{array} \quad \begin{array}{c|c} \boxed{2} & \boxed{3} \\ \hline 1 & \end{array} \quad \begin{array}{c|c} \boxed{3} & \\ \hline 1 & \boxed{2} \end{array} \quad \begin{array}{c} 3 \\ \hline 2 \\ \hline 1 \end{array} \quad (3.12)$$

We have filled in the boxes with their *contents* 1 2 3 in order from southwest to northeast. Reading contents along rows starting from the top row, we obtain a distinguished set $\{123, 231, 312, 321\}$ of representatives for the \sim -equivalence classes of (homogeneous) words in $\langle I \rangle_\alpha$. The corresponding irreducible representations $L(123)$, $L(231)$, $L(312)$ and $L(321)$ have characters 123, 231 + 213, 312 + 132 and 321, respectively. They give all of the irreducible graded H_α -modules up to degree shift. Their projective covers $P(123)$, $P(231)$, $P(312)$ and $P(321)$ can be obtained explicitly as the left ideals generated by the idempotents 1_{123} , 1_{231} , 1_{312} and 1_{321} , respectively. This is so explicit that one can then compute the endomorphism algebra of the resulting minimal projective generator $P(123) \oplus P(231) \oplus P(312) \oplus P(321)$ directly, to see that it is isomorphic to the tensor product $A \otimes A \otimes \mathbb{K}[x]$ where A is the path algebra of the quiver (3.11) as above. This is a graded algebra of global dimension 3, and H_α is graded Morita equivalent to it, so H_α has global dimension 3 too.

Theorem 3.13 (Brundan-Kleshchev). *Suppose the graph underlying the quiver is the Dynkin diagram A_n and that $\alpha = \alpha_1 + \cdots + \alpha_n$ is the highest root. Then all irreducible graded H_α -modules are homogeneous and are parametrized naturally by the skew-hooks with n boxes as in the example above. Moreover H_α is graded Morita equivalent to $A^{\otimes(n-1)} \otimes \mathbb{K}[x]$, which is of global dimension n .*

In fact, as we'll discuss in more detail later on, all the quiver Hecke algebras whose underlying graph is a finite type Dynkin diagram are of finite global dimension. We expect the converse of this statement holds too: it should be the case that H_α has finite global dimension for all $\alpha \in Q^+$ if and only if the underlying graph is a finite type Dynkin diagram.

We end with one example of infinite global dimension: suppose the underlying graph is $0=1$, that is, the affine Dynkin diagram $A_1^{(1)}$. The words 01 and 10 are homogeneous, so we have the one-dimensional homogeneous H_δ -modules $L(01)$ and $L(10)$, where $\delta = \alpha_0 + \alpha_1$ is the smallest imaginary root. Working in $\mathbf{f} = \mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Rep}(H)]$, we have that $\theta_i = [L(i)]/(1 - q^2)$. Hence

$$\begin{aligned}\theta_{01} &= [L(0) \circ L(1)]/(1 - q^2)^2 = ([L(01)] + q^2[L(10)])/(1 - q^2)^2, \\ \theta_{10} &= [L(1) \circ L(0)]/(1 - q^2)^2 = ([L(10)] + q^2[L(01)])/(1 - q^2)^2.\end{aligned}$$

We deduce that $[L(01)] = \frac{1-q^2}{1+q^2}(\theta_{01} - q^2\theta_{10})$. Using also (3.8), we can then compute the inner product

$$([L(01)], [L(01)]) = \frac{1-q^2}{1+q^2}(\theta_{01} - q^2\theta_{10}, [L(01)]) = \frac{1-q^2}{1+q^2} \notin \mathbb{Z}[q, q^{-1}].$$

On the other hand, if H_δ has finite global dimension, we can find a finite projective resolution $P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow L(01) \rightarrow 0$, to deduce that

$$([L(01)], [L(01)]) = \sum_{i=0}^n (-1)^i ([P_i], [L(01)]) \in \mathbb{Z}[q, q^{-1}].$$

This contradiction establishes that H_δ has infinite global dimension.

Notes. The main categorification theorem (Theorem 3.9) is [18, Theorem 1.1], and our exposition of the proof is based closely on the original account there. The linear independence of characters (Theorem 3.3) is [18, Theorem 3.17]; the proof given there is essentially the same as the proof of the analogous result for degenerate affine Hecke algebras in Kleshchev's book [21, Theorem 5.3.1] which in turn repeated an argument written down by Vazirani in her thesis based on the results of [11]; in the context of affine Hecke algebras this result goes back to Bernstein. The second equality in (3.2) is justified in [23, Lemma 3.2]. The Mackey Theorem (Theorem 3.6) is [18, Proposition 2.18]. The categorification of the Serre relations (Lemma 3.10) was first worked out in a slightly weaker form in [19, Corollary 7], and in the form described here in [30, Lemma 3.13].

The identification of the canonical and dual canonical bases (Theorem 3.11) is [31, Corollary 5.8] or [32, Theorem 4.5]. The proof of this theorem depends on the geometric realization of quiver Hecke algebras, hence is valid only for the case of symmetric Cartan matrices. If \mathbb{K} is of positive characteristic then the bases \mathbf{B} and \mathbf{B}^* arising from the quiver Hecke algebras are different in general from the canonical and dual canonical bases. For a while there was a conjecture formulated by Kleshchev and Ram [23, Conjecture 7.3] asserting that they should be the same independent of characteristic at least in all finite ADE types, but this turned out to be false. Various counterexamples are explained by Williamson in [38]; see also [6, Example 2.16]. Note though that \mathbf{B}^* is always a *perfect basis* in the sense of Berenstein-Kazhdan [3].

The classification of homogeneous representations (Theorem 3.12) was worked out by Kleshchev and Ram in [22, Theorem 3.4]. The version proved here is a slight generalization since we include quivers with multiple edges. In [22, Theorem 3.10], Kleshchev and Ram go on to introduce a special class of “minuscule” homogeneous representations, showing that the dimensions of these representations are given by the Petersen-Proctor hook formula, thus revealing another unexpected connection to combinatorics.

Theorem 3.13 is unpublished but makes a good exercise!

4. Finite type

In the final section we restrict our attention to finite ADE types, i.e. we assume that the graph underlying the quiver is a finite ADE Dynkin diagram.

PBW and dual PBW bases. Since we are now in finite ADE type, the underlying Kac-Moody algebra \mathfrak{g} is actually a finite dimensional simple Lie algebra. Let W denote the (finite) Weyl group associated to \mathfrak{g} , which is the subgroup of $GL(P)$ generated by the *simple reflections* $\{s_i \mid i \in I\}$ defined from $s_i(\lambda) = \lambda - (\alpha_i \cdot \lambda)\alpha_i$. Let $R := \bigcup_{i \in I} W(\alpha_i) \subset Q$ be the set of *roots* and $R^+ := R \cap \bigoplus_{i \in I} \mathbb{N}\alpha_i$ be the *positive roots*. We fix once and for all a reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$ for the longest word $w_0 \in W$ (so $N = |R^+|$). There is a corresponding *convex order* \prec on R^+ defined from $\alpha_{i_1} \prec s_{i_1}(\alpha_{i_2}) \prec \cdots \prec s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})$. By a *Kostant partition* of $\alpha \in Q^+$, we mean a sequence $\lambda = (\lambda_1, \dots, \lambda_l)$ of positive roots summing to α such that $\lambda_1 \succeq \cdots \succeq \lambda_l$ with respect to this fixed convex order. Let $\text{KP}(\alpha)$ denote the set of all Kostant partitions of α and $\text{KP} := \bigcup_{\alpha \in Q^+} \text{KP}(\alpha)$.

Associated to the reduced expression/convex order just fixed, Lusztig has introduced a *PBW basis* for \mathbf{f} , indexed as it should be by KP . To construct this, Lusztig first defines *root vectors* $\{r_\alpha \mid \alpha \in R^+\}$. Then the *PBW monomial* associated to $\lambda = (\lambda_1, \dots, \lambda_l) \in \text{KP}$ is defined from

$$r_\lambda := r_{\lambda_1} \cdots r_{\lambda_l} / [\lambda]!, \quad (4.1)$$

where $[\lambda]! := \prod_{\beta \in R^+} m_\beta(\lambda)$, and $m_\beta(\lambda)$ denotes the multiplicity of β in λ . In Lusztig’s approach, the definition of the root vector r_α depends on a certain action of the braid group $\langle T_i \mid i \in I \rangle$ associated to W on the full quantized enveloping algebra $U_q(\mathfrak{g})$: fixing an embedding $\mathbf{f} \hookrightarrow U_q(\mathfrak{g})$ one sets $r_\alpha := T_{i_1} \cdots T_{i_{r-1}}(\theta_{i_r})$ if $\alpha = s_{i_1} \cdots s_{i_{r-1}}(\alpha_{i_r})$. We skip the precise details here because there is also a more elementary recursive approach to the definition of the root vector r_α , well known in type A but only recently established in full generality in types D and E. To formulate this we need the notion of a *minimal pair* for $\alpha \in R^+$: a pair (β, γ) of positive roots such that $\beta \succ \gamma$, $\beta + \gamma = \alpha$, and there is no other pair (β', γ') of positive roots with $\beta' + \gamma' = \alpha$ and $\beta \succ \beta' \succ \alpha \succ \gamma' \succ \gamma$.

Lemma 4.1. *Let $\alpha \in R^+$. If $\alpha = \alpha_i$ for some $i \in I$ then $r_\alpha = \theta_i$. If α is not simple then $r_\alpha = r_\gamma r_\beta - qr_\beta r_\gamma$ for any minimal pair (β, γ) for α .*

Proof. This follows from Theorem 4.8 below. \square

Also define the *dual root vectors* $\{r_\alpha^* \mid \alpha \in R^+\}$ by setting $r_\alpha^* := (1 - q^2)r_\alpha$. Then for $\lambda = (\lambda_1, \dots, \lambda_l) \in \text{KP}$ define the *dual PBW monomial*

$$r_\lambda^* := q^{s_\lambda} r_{\lambda_1}^* \cdots r_{\lambda_l}^* \quad (4.2)$$

where $s_\lambda := \frac{1}{2} \sum_{\beta \in R^+} m_\beta(\lambda)(m_\beta(\lambda) - 1)$. Lusztig's fundamental result about PBW and dual PBW bases is as follows.

Theorem 4.2 (Lusztig). *The sets $\{r_\lambda \mid \lambda \in \text{KP}\}$ and $\{r_\lambda^* \mid \lambda \in \text{KP}\}$ give bases for $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]}^*$ and $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]}^*$, respectively. Moreover these two bases are dual with respect to the pairing (\cdot, \cdot) .*

For a Kostant partition $\lambda = (\lambda_1, \dots, \lambda_l)$, we set $\lambda'_k := \lambda_{l+1-k}$ for $k = 1, \dots, l$. Then introduce a partial order \preceq on KP by declaring that $\lambda \prec \mu$ if and only if both of the following hold:

- ▷ $\lambda_1 = \mu_1, \dots, \lambda_{k-1} = \mu_{k-1}$ and $\lambda_k \prec \mu_k$ for some k such that λ_k and μ_k both make sense;
- ▷ $\lambda'_1 = \mu'_1, \dots, \lambda'_{k-1} = \mu'_{k-1}$ and $\lambda'_k \succ \mu'_k$ for some k such that λ'_k and μ'_k both make sense.

We note for $\alpha \in R^+$ that the unique smallest element of $\text{KP}(\alpha)$ is the one-part Kostant partition (α) , and the next smallest elements are the minimal pairs (β, γ) defined above. The important point is that the bar involutions \mathbf{b} and \mathbf{b}^* act on the PBW and dual PBW bases in a triangular way with respect to this order:

$$\mathbf{b}(r_\lambda) = r_\lambda + (\text{a } \mathbb{Z}[q, q^{-1}]\text{-linear combination of } r_\mu \text{ for } \mu \succ \lambda), \quad (4.3)$$

$$\mathbf{b}^*(r_\lambda^*) = r_\lambda^* + (\text{a } \mathbb{Z}[q, q^{-1}]\text{-linear combination of } r_\mu^* \text{ for } \mu \prec \lambda). \quad (4.4)$$

Combined with Lusztig's lemma, this triangularity implies the existence of unique bases $\{b_\lambda \mid \lambda \in \text{KP}\}$ and $\{b_\lambda^* \mid \lambda \in \text{KP}\}$ for $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]}^*$ and $\mathbf{f}_{\mathbb{Z}[q, q^{-1}]}^*$, respectively, such that

$$\mathbf{b}(b_\lambda) = b_\lambda, \quad b_\lambda = r_\lambda + (\text{a } q\mathbb{Z}[q]\text{-linear combination of } r_\mu \text{ for } \mu \succ \lambda), \quad (4.5)$$

$$\mathbf{b}^*(b_\lambda^*) = b_\lambda^*, \quad b_\lambda^* = r_\lambda^* + (\text{a } q\mathbb{Z}[q]\text{-linear combination of } r_\mu^* \text{ for } \mu \prec \lambda). \quad (4.6)$$

The basis $\{b_\lambda \mid \lambda \in \text{KP}\}$ is the Lusztig-Kashiwara canonical basis. The following lemma shows that $\{b_\lambda^* \mid \lambda \in \text{KP}\}$ is the dual canonical basis.

Lemma 4.3. *For $\lambda, \mu \in \text{KP}$ we have that $(b_\lambda, b_\mu^*) = \delta_{\lambda, \mu}$.*

Proof. By (4.5)–(4.6) and the duality of the PBW and dual PBW basis vectors, we have that $(b_\lambda, b_\mu^*) \in \delta_{\lambda, \mu} + q\mathbb{Z}[q]$. Now it remains to observe that it is bar-invariant:

$$(b_\lambda, b_\mu^*) = (\mathbf{b}(b_\lambda), b_\mu^*) = \overline{(\mathbf{b}(b_\lambda), b_\mu^*)} = \overline{(b_\lambda, b_\mu^*)}.$$

\square

Proper standard modules. If $\lambda = (\alpha)$ for $\alpha \in R^+$ then it is minimal in $\text{KP}(\alpha)$, hence by (4.6) the dual canonical basis element b_λ^* must simply be equal to the dual root element r_α^* . For \mathbb{K} of characteristic 0, we deduce immediately from this and the geometric Theorem 3.11 that for each $\alpha \in R^+$ there is a (unique up to isomorphism) irreducible H -module $L(\alpha)$ such that $[L(\alpha)] = r_\alpha^*$. These modules are known as *cuspidal modules*. McNamara has recently found an elementary inductive proof of the existence of cuspidal modules based just on Theorem 3.9, which works for ground fields \mathbb{K} of positive characteristic too. In fact in many cases, including all positive roots and all convex orders in type A, the cuspidal module $L(\alpha)$ turns out to be homogeneous, so can be constructed explicitly via Theorem 3.12.

Now suppose we are given $\alpha \in Q^+$ and $\lambda = (\lambda_1, \dots, \lambda_l) \in \text{KP}(\alpha)$. Define the *proper standard module*

$$\bar{\Delta}(\lambda) := q^{s_\lambda} L(\lambda_1) \circ \cdots \circ L(\lambda_l). \quad (4.7)$$

Comparing with the definition (4.2), we have that $[\bar{\Delta}(\lambda)] = r_\lambda^*$, i.e. proper standard modules categorify the dual PBW basis. Then let

$$L(\lambda) := \bar{\Delta}(\lambda) / \text{rad } \bar{\Delta}(\lambda).$$

Now we can state the following classification of irreducible H -modules.

Theorem 4.4 (Kleshchev-Ram, Kato, McNamara). *The modules $\{L(\lambda) \mid \lambda \in \text{KP}\}$ give a complete set of pairwise inequivalent \circledast -self-dual irreducible H -modules. Moreover, all composition factors of $\text{rad } \bar{\Delta}(\lambda)$ are of the form $q^d L(\mu)$ for $\mu \prec \lambda$ and $d \in \mathbb{Z}$.*

Theorem 4.4 can be viewed as a vast generalization of Zelevinsky's classification of irreducible representations of affine Hecke algebras via "multisegments." Zelevinsky's result can be interpreted as treating the case that the Dynkin diagram is $1-2-\cdots-n$, so the positive roots are $\alpha_{i,j} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for $1 \leq i \leq j \leq n$, and the convex order is defined by $\alpha_{i,j} \prec \alpha_{k,l}$ if and only if $i < k$ or $i = k$ and $j < l$. The cuspidal module $L(\alpha_{i,j})$ is the one-dimensional homogeneous module corresponding to the homogeneous word $[i, j] := i(i+1)\cdots j$. We call this a *segment*. Then a Kostant partition $\lambda = (\alpha_{i_1,j_1}, \dots, \alpha_{i_r,j_r})$ is the same thing as a *multisegment*, i.e. a non-increasing sum of segments $[i_1, j_1] + \cdots + [i_r, j_r]$. The proper standard module $\bar{\Delta}(\lambda)$ has character $q^{s_\lambda} [i_1, j_1] \circ \cdots \circ [i_r, j_r]$ obtained by taking the shuffle product of these segments. The irreducible heads of these modules give all the irreducible H -modules up to isomorphism and degree shift. For example if $\alpha = \alpha_{1,3}$ then the irreducible graded H_α -modules are indexed by $\text{KP}(\alpha) = \{(\alpha_{1,3}), (\alpha_{2,3}, \alpha_1), (\alpha_3, \alpha_{1,2}), (\alpha_3, \alpha_2, \alpha_1)\}$. These modules are the homogeneous representations parametrized by the skew-hooks from (3.12). To translate from skew-hook to Kostant partition, record the contents in the rows of the skew-hook from top to bottom to obtain the corresponding multisegment.

If \mathbb{K} is of characteristic zero then we have that $[\bar{\Delta}(\lambda)] = r_\lambda^*$ and $[L(\mu)] = b_\mu^*$ thanks to Theorem 3.11. Thus the coefficients $p_{\lambda,\mu}(q)$ defined from the equivalent

expansions

$$r_\lambda^* = \sum_{\mu \in \text{KP}} p_{\lambda, \mu}(q) b_\mu^*, \quad b_\mu = \sum_{\lambda \in \text{KP}} p_{\lambda, \mu}(q) r_\lambda$$

compute the graded composition multiplicities $[\bar{\Delta}(\lambda) : L(\mu)]$. Thus Theorem 4.4 has the following application.

Corollary 4.5. *We have that $p_{\lambda, \mu}(q) = 0$ unless $\mu \preceq \lambda$, $p_{\lambda, \lambda}(q) = 1$, and $p_{\lambda, \mu}(q) \in q\mathbb{N}[q]$ if $\mu \prec \lambda$.*

Finally we record a lemma which is at the heart of McNamara's inductive proof of the existence of cuspidal modules and his approach to Theorem 4.4. We will apply this several times also in the next subsection.

Lemma 4.6 ("McNamara's Lemma"). *Suppose we are given $\alpha \in R^+$ and $\beta, \gamma \in Q^+$ with $\beta + \gamma = \alpha$. If $\text{Res}_{\beta, \gamma}^\alpha L(\alpha) \neq 0$ then β is a sum of positive roots $\preceq \alpha$ and γ is a sum of positive roots $\succeq \alpha$.*

Proof. Forget about the grading throughout this proof. We just explain how to show that β is a sum of positive roots $\preceq \alpha$; the statement about γ follows similarly. In the next paragraph, we'll prove the following claim:

Claim. *If $\text{Res}_{\beta, \gamma}^\alpha L(\alpha) \neq 0$ for $\beta \in R^+$, $\gamma \in Q^+$ with $\beta + \gamma = \alpha$ then $\beta \preceq \alpha$.*

Now suppose that we are in the situation of the lemma for non-zero β, γ . Let $L(\lambda) \boxtimes L(\mu)$ be a composition factor of $\text{Res}_{\beta, \gamma}^\alpha L(\alpha)$ for some $\lambda \in \text{KP}(\beta)$ and $\mu \in \text{KP}(\gamma)$. Then it's clear that $\text{Res}_{\lambda_1, \alpha - \lambda_1}^\alpha L(\alpha)$ is non-zero. Invoking the claim, we deduce that $\lambda_1 \preceq \alpha$. Hence the parts of λ are positive roots $\preceq \alpha$ summing to β , and we are done.

To prove the claim, suppose for a contradiction that it is false, and look at the counterexample in which β is maximal; of course we have that $\beta \succ \alpha$ and $\gamma \neq 0$. Let $L(\lambda) \boxtimes L(\mu)$ be a submodule of $\text{Res}_{\beta, \gamma}^\alpha L(\alpha)$ for $\lambda \in \text{KP}(\beta), \mu \in \text{KP}(\gamma)$. As in the previous paragraph, we have that $\text{Res}_{\lambda_1, \alpha - \lambda_1}^\alpha L(\alpha) \neq 0$, so by the maximality of β we must have that $\lambda_1 \preceq \beta$. Since (β) is minimal in $\text{KP}(\beta)$ this implies that $\lambda = (\beta)$. We have shown that $L(\beta) \boxtimes L(\mu)$ is a submodule of $\text{Res}_{\beta, \gamma}^\alpha L(\alpha)$ for some $\mu \in \text{KP}(\gamma)$. By Frobenius reciprocity, we get a non-zero homomorphism $L(\beta) \circ L(\mu) \rightarrow L(\alpha)$. If $\beta \succeq \mu_1$ then $L(\beta) \circ L(\mu)$ is a quotient of $\bar{\Delta}(\beta \sqcup \mu)$, where we write $\beta \sqcup \mu$ for the Kostant partition obtained from μ by adding the root β to the beginning. By Theorem 4.4 we know that $\bar{\Delta}(\beta \sqcup \mu)$ has irreducible head $L(\beta \sqcup \mu)$, and we deduce that $L(\beta \sqcup \mu) \cong L(\alpha)$. This shows that $\beta \sqcup \mu = (\alpha)$, hence $\beta = \alpha$, contradicting $\beta \succ \alpha$.

So we have that $\beta \prec \mu_1$. Let $\bar{\mu} \in \text{KP}(\gamma - \mu_1)$ be the (possibly empty) Kostant partition obtained from μ by removing its first part, so $\mu = \mu_1 \sqcup \bar{\mu}$. Since $L(\mu)$ is a quotient of $L(\mu_1) \circ \bar{\Delta}(\bar{\mu})$, Frobenius reciprocity gives us a non-zero homomorphism $(L(\beta) \circ L(\mu_1)) \boxtimes \bar{\Delta}(\bar{\mu}) \rightarrow \text{Res}_{\beta + \mu_1, \gamma - \mu_1}^\alpha L(\alpha)$. Hence there's some $\nu \in \text{KP}(\beta + \mu_1)$ such that $L(\nu) \boxtimes L(\bar{\mu})$ is a composition factor of $\text{Res}_{\beta + \mu_1, \gamma - \mu_1}^\alpha L(\alpha)$. As above, the maximality of β implies that $\nu_1 \preceq \beta$, hence all parts of ν are $\preceq \beta$. The parts of ν sum to $\beta + \mu_1$, so we deduce that $\beta + \mu_1$ is a sum of roots $\preceq \beta$, as well as already being a sum of β and another root $\mu_1 \succ \beta$. This contradicts a general property of convex orders (e.g. see [6, Lemma 2.4]). \square

Standard modules and homological algebra. Just as the proper standard modules $\bar{\Delta}(\lambda)$ categorify the dual PBW basis vectors r_λ^* , there are *standard modules* $\Delta(\lambda)$ which categorify the PBW basis vectors r_λ . The definition of these parallels the definition of the PBW basis elements: we first define *root modules* $\Delta(\alpha)$ for each $\alpha \in R^+$ so that $[\Delta(\alpha)] = r_\alpha$, then introduce *divided power modules* $\Delta(\alpha^m)$ for $\alpha \in R^+$ and $m \geq 1$ so that $[\Delta(\alpha^m)] = r_\alpha^m/[m]!$, then finally set

$$\Delta(\lambda) := \Delta(\mu_1^{m_1}) \circ \cdots \circ \Delta(\mu_k^{m_k}) \quad (4.8)$$

for a Kostant partition λ written as $\lambda = (\mu_1^{m_1}, \dots, \mu_k^{m_k})$ for $\mu_1 \succ \cdots \succ \mu_k$. It is then automatic from (4.1) that $[\Delta(\lambda)] = r_\lambda$.

Perhaps the easiest way to define the root module $\Delta(\alpha)$ is by inducing the cuspidal module $L(\alpha)$ from the previous subsection (although this only works in simply-laced types). Given $\alpha \in R^+$ of height n , let H'_α be the subalgebra of H_α generated by the elements $\{1_i, \tau_j, x_j - x_{j+1} \mid i \in \langle I \rangle_\alpha, 1 \leq j \leq n-1\}$. Then set

$$\Delta(\alpha) := H_\alpha \otimes_{H'_\alpha} L(\alpha). \quad (4.9)$$

Recalling (2.10), we can choose the special word \mathbf{i} there so that i_1 appears with a multiplicity k which is non-zero in the field \mathbb{K} ; this is always possible by inspection of the ADE root systems. Then, letting $z := z_1 + \cdots + z_k \in Z(H_\alpha)_2$, we have that $H_\alpha = H'_\alpha \otimes \mathbb{K}[z]$ as an algebra. Hence we have that $\Delta(\alpha) = L(\alpha) \boxtimes \mathbb{K}[z]$ as a module, i.e. it is an infinite self-extension of copies of $L(\alpha)$. The following is obvious from this description combined with Schur's Lemma.

Lemma 4.7. *The root module $\Delta(\alpha)$ has irreducible head $L(\alpha)$, and we have that $[\Delta(\alpha)] = [L(\alpha)]/(1 - q^2)$ in the Grothendieck group, hence $[\Delta(\alpha)] = r_\alpha$. Moreover $\text{End}_{H_\alpha}(\Delta(\alpha)) = \mathbb{K}[z]$.*

There is also a quite different recursive description of $\Delta(\alpha)$ which is essentially a categorification of Lemma 4.1. First if $\alpha = \alpha_i$ for $i \in I$ then we have that $\Delta(\alpha) \cong H_{\alpha_i}$. Then if $\alpha \in R^+$ is of height at least two, we pick a minimal pair (β, γ) for α . As $[\Delta(\beta)] = (1 + q^2 + q^4 + \cdots)[L(\beta)]$, all composition factors of $\Delta(\beta)$ are isomorphic to $L(\beta)$ up to a degree shift. Similarly all composition factors of $\Delta(\gamma)$ are isomorphic to $L(\gamma)$ up to a degree shift. Now an application of McNamara's Lemma reveals that there is only one non-zero layer in the Mackey filtration of $\text{Res}_{\beta, \gamma}^\alpha \Delta(\gamma) \circ \Delta(\beta)$ from Theorem 3.6, and this layer is isomorphic to $q\Delta(\beta) \boxtimes \Delta(\gamma)$. Hence $\text{Res}_{\beta, \gamma}^\alpha \Delta(\gamma) \circ \Delta(\beta) \cong q\Delta(\beta) \boxtimes \Delta(\gamma)$. Frobenius reciprocity then gives us a canonical homomorphism

$$\phi : q\Delta(\beta) \circ \Delta(\gamma) \rightarrow \Delta(\gamma) \circ \Delta(\beta), \quad 1_\alpha \otimes (v_1 \otimes v_2) \mapsto \tau_w \otimes (v_2 \otimes v_1)$$

where w is the permutation $(1, \dots, m+n) \mapsto (n+1, \dots, m+n, 1, \dots, n)$ for $m := \text{ht}(\beta)$ and $n := \text{ht}(\gamma)$.

Theorem 4.8 (Brundan-Kleshchev-McNamara). *The homomorphism ϕ just defined is injective and its cokernel is isomorphic to the root module $\Delta(\alpha)$ from (4.9). Hence there is a short exact sequence*

$$0 \longrightarrow q\Delta(\beta) \circ \Delta(\gamma) \xrightarrow{\phi} \Delta(\gamma) \circ \Delta(\beta) \longrightarrow \Delta(\alpha) \longrightarrow 0.$$

The proof of Theorem 4.8 depends in a crucial way on some homological algebra: it requires knowing in advance that

$$\mathrm{Ext}_{H_\alpha}^1(L(\alpha), L(\alpha)) \cong q^{-2}\mathbb{K}, \quad \mathrm{Ext}_{H_\alpha}^i(L(\alpha), L(\alpha)) = 0 \text{ for } i \geq 2. \quad (4.10)$$

This is surprisingly tricky to prove. McNamara found a proof assuming the finiteness of global dimension of H_α , and another way based on some explicit computations which can be made just for certain special choices of the convex order (for which almost all $L(\alpha)$ are homogeneous). He also showed how to deduce the finiteness of global dimension from (4.10). Thus one first proves (4.10) for a particular convex order, then deduces the finiteness of global dimension, then from that extends (4.10) to arbitrary convex orders, and finally uses that to prove Theorem 4.8.

Next we explain how the divided powers $\Delta(\alpha^m)$ are defined. Using the decomposition $\Delta(\alpha) = L(\alpha) \boxtimes \mathbb{K}[z]$, it is easy to deduce from (4.10) that

$$\mathrm{Ext}_{H_\alpha}^i(\Delta(\alpha), \Delta(\alpha)) = \mathrm{Ext}_{H_\alpha}^i(\Delta(\alpha), L(\alpha)) = 0 \text{ for } i \geq 1. \quad (4.11)$$

Using McNamara's Lemma once again, one checks that the Mackey filtration of $\mathrm{Res}_{\alpha, \alpha}^{2\alpha} \Delta(\alpha) \circ \Delta(\alpha)$ has exactly two non-zero layers, $\Delta(\alpha) \boxtimes \Delta(\alpha)$ at the bottom and $q^{-2}\Delta(\alpha) \boxtimes \Delta(\alpha)$ at the top. Then (4.11) implies that this extension splits, hence there is a non-zero homogeneous homomorphism $\Delta(\alpha) \boxtimes \Delta(\alpha) \rightarrow \mathrm{Res}_{\alpha, \alpha}^{2\alpha} \Delta(\alpha) \circ \Delta(\alpha)$ of degree -2 . Applying Frobenius reciprocity we get from this a canonical degree -2 endomorphism

$$\tau : \Delta(\alpha) \circ \Delta(\alpha) \rightarrow \Delta(\alpha) \circ \Delta(\alpha), \quad 1_{2\alpha} \otimes (v_1 \otimes v_2) \mapsto \tau_w \otimes (v_2 \otimes v_1),$$

where $w : (1, \dots, 2n) \mapsto (n+1, \dots, 2n, 1, \dots, n)$. Recalling Lemma 4.7, one then shows that $\Delta(\alpha)$ has a *unique* degree 2 endomorphism x such that $\tau \circ (1x) = (x1) \circ \tau + 1$ on $\Delta(\alpha) \circ \Delta(\alpha)$. Then setting $\tau_i := 1^{i-1} \tau 1^{m-i-1}$ and $x_i := 1^{i-1} x 1^{m-i}$ we obtain a right action of the nil Hecke algebra NH_m on $\Delta(\alpha)^{\circ m}$. Finally, recalling the idempotent $e_m \in NH_m$ from (2.4), we set

$$\Delta(\alpha^m) := q^{\frac{1}{2}m(m-1)} \Delta(\alpha)^{\circ m} e_m, \quad (4.12)$$

and get from Corollary 2.4 that $[\Delta(\alpha)^{\circ m}] = [m]![\Delta(\alpha^m)]$. Hence we have indeed categorified the divided power $r_\alpha^m/[m]!$

This completes our sketch of the definition of the standard modules $\Delta(\lambda)$. The following theorem collects various homological properties of the finite type quiver Hecke algebras. Most of these are deduced from the definition (4.8), either using generalized Frobenius reciprocity together with (4.10)–(4.11) or by arguing inductively via Theorem 4.8. They are reminiscent of the homological properties of a quasi-hereditary algebra.

Theorem 4.9 (Kato, McNamara, Brundan-Kleshchev-McNamara). *Suppose we are given α of height n and $\lambda, \mu \in \mathrm{KP}(\alpha)$.*

- (1) *The standard module $\Delta(\lambda)$ has an exhaustive descending filtration with $\bar{\Delta}(\lambda)$ at the top and all other sections of the form $q^{2d}\bar{\Delta}(\lambda)$ for $d > 0$.*

- (2) We have that $\text{Ext}_{H_\alpha}^i(\Delta(\lambda), \Delta(\lambda)) = \text{Ext}_{H_\alpha}^i(\Delta(\lambda), \bar{\Delta}(\lambda)) = 0$ for $i \geq 1$.
- (3) Setting $\bar{\nabla}(\mu) := \bar{\Delta}(\mu)^*$, we have that $\text{Dim Ext}_{H_\alpha}^i(\Delta(\lambda), \bar{\nabla}(\mu)) = \delta_{i,0}\delta_{\lambda,\mu}$ for $i \geq 0$.
- (4) The projective dimension of $\Delta(\lambda)$ is bounded above by $n - l$, where l is the number of parts of λ .
- (5) The global dimension of H_α is equal to n .

Corollary 4.10. For $\lambda \in \text{KP}$, let $P(\lambda)$ be the projective cover of $L(\lambda)$. Then $P(\lambda)$ has a finite filtration with sections of the form $\Delta(\mu)$ (up to a degree shift) and graded multiplicities satisfying $[P(\lambda) : \Delta(\mu)] = [\bar{\Delta}(\mu) : L(\lambda)]$.

Corollary 4.11. For any $\lambda \in \text{KP}$, we have that

$$\Delta(\lambda) \cong P(\lambda) \left/ \sum_{\mu \preceq \lambda} \sum_{f: P(\mu) \rightarrow P(\lambda)} \text{im } f \right., \quad \bar{\Delta}(\lambda) \cong P(\lambda) \left/ \sum_{\mu \not\preceq \lambda} \sum_{f: P(\mu) \rightarrow \text{rad } P(\lambda)} \text{im } f \right.,$$

summing over all (not necessarily homogeneous) homomorphisms f .

Notes. The key facts about the algebra \mathbf{f} summarized here are all proved in Lusztig's book [26]. The "bilexicographic" partial order \preceq was introduced originally in [27, §3]. The triangularity (4.4) of the bar involution on the dual PBW basis follows from Theorem 4.4; more direct proofs avoiding quiver Hecke algebras also exist.

Cuspidal modules and the definition (4.7) arose first in the work of Kleshchev and Ram [23], which only treats certain rather special convex orders as in [25]. Kato then gave a completely different approach to the construction of cuspidal modules working in the framework of [32], and based on the geometric definition of so-called *Saito reflection functors* which categorify Lusztig's braid group action. McNamara subsequently found an elegant algebraic treatment which can be found in [27, Theorem 3.1]. The positivity of the coefficients of $p_{\lambda,\mu}(q)$ from Corollary 4.5 was conjectured a long time ago by Lusztig and proved for the first time in full generality by Kato in [17]. It is one of the first genuine applications of quiver Hecke algebras to the theory of canonical bases.

A geometric approach to the definition of standard modules can be found in [17]. The construction explained here is based instead on our article [6], which also covers the non-simply-laced finite types BCFG; non-simply-laced types are more complicated as we cannot exploit the naive approach of inducing from H'_α . The homological properties in Theorem 4.9 are proved in [17], [27] and [6]. In particular the finiteness of global dimension for simply-laced types in characteristic zero was established originally by Kato; in type A it can also be deduced from [29]. The exact value for the global dimension of H_α was determined later by McNamara; in type A this follows also from Theorem 3.13.

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